DISCRETE SYSTEM SENSITIVITY AND VARIABLE INCREMENT OPTIMAL SAMPLING

By

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Discrete system sensitivity is investigated and a scheme presented for the optimal adjustment of the sampling rate of a sampled-data system. As background for the sensitivity study, a survey of the historical development of sensitivity analysis is presented. The survey includes recent developments toward a generalized approach to sensitivity analysis. Also included are some of the applications of sensitivity to optimal control.

The investigation of discrete system sensitivity includes variations in system parameters and sampling intervals. Two approaches to parameter sensitivity are used. One method makes use of a perturbation matrix and is only applicable to linear systems. The general approach, using partial derivatives, is useful in linear and nonlinear systems. Discrete sensitivity equations are derived for both methods.

Sampling interval sensitivity is investigated for global and local effects. For the global sensitivity function, all sampling intervals are equal and undergo the same variation. In local sampling interval sensitivity, only one interval is assured to change.

A method for the optimal adjustment of the sampling rate of a sampled-data system is presented. The technique uses local and error sampling interval sensitivity in sampling interval formulas that constrain the magnitude of the reconstruction and modeling errors of the discrete system response. The resulting sample adjustment scheme is suitable for real-time digital computer simulation. The possibility of using the sampling interval adjustment scheme for optimal control is discussed. Several example problems are used to illustrate the technique. In each case, variable increment sampling improved sampling efficiency and computer utilization.

CHAPTER 1

INTRODUCTION

Sensitivity Analysis In Automatic Control

A general definition of sensitivity analysis states that it is the development and use of equations for the partial derivatives of the system response with respect to system parameters [1].[1] Sensitivity was probably first applied to control systems by Bode [2] in 1945. From 1945 until about 1960, only a few people were interested in sensitivity theory and thus advances occurred rather slowly. However, the advent of adaptive control and its associated problems of identification and parameter adjustment brought a renewed interest in sensitivity. Interest in sensitivity was also stimulated by the need to know more about system dynamics and the effect of perturbations.

In the last few years, considerable progress has been made concerning the theory of sensitivity analysis. It now includes a wide variety of perturbations and has been extended to optimal control. It is also being used much earlier in the design process. The future of sensitivity

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[1] Bracketed numbers refer to the list of references collected at the end of each chapter.

appears to be very promising, particularly in discrete and hybrid systems where work is just beginning.

Research Objectives

This dissertation has three basic goals. The first objective is to survey the existing applications of sensitivity to control systems. The second aim is to develop sensitivity procedures for discrete systems. The third objective is to use the discrete sensitivity techniques that have been developed to implement variable increment optimal sampling.

The project is computer oriented and the aim is to develop algorithms for various sensitivity functions that are convenient for digital simulation. All of the techniques will be illustrated and recommendations regarding accuracy and convenience will be made.

Plan of the Dissertation

Chapter 1 is an introduction to the dissertation. It includes a brief discussion of sensitivity analysis and outlines the objectives of the research. Chapter 1 also presents a summary of each chapter and gives the notation to be used throughout the dissertation.

Chapter 2 surveys the existing work on sensitivity analysis. A brief historical development is presented first as an introduction and background for a more

detailed look at sensitivity in optimal control. The various approaches that have been suggested are presented and their uses and interrelations are discussed.

In Chapter 3, the application of sensitivity to discrete systems is investigated. Discrete sensitivity equations are developed for parameter and sampling interval sensitivity. The chapter also includes example problems.

Chapter 4 illustrates the application of techniques developed in Chapter 3. Sampling interval sensitivity is used to implement variable increment optimal sampling. The use of variable increment sampling in optimal control is also discussed. Several example problems are also included.

Chapter 5 contains the conclusions and recommendations for discrete sensitivity and variable increment sampling.

Notation

Throughout this dissertation, vector-matrix notation is used to represent system dynamics. Scalars are indicated by lower case Greek or Roman letters. The only exception is the performance index J of an optimal control system. Column vectors are indicated by underlined, lower case Greek or Roman letters; e.g., $\underline{x}$. Capital Greek or Roman letters are used to denote matrices.

Subscripts have a number of uses. Two subscripts indicate the row and column of a component of a matrix;

e.g., x_{ij}. A single subscript is used to denote the component of a vector; e.g., x_i. A single subscript also means a particular column of a matrix; e.g., $\underline{v}_i$. In discrete systems, subscripts also indicate a function during a particular sampling interval; e.g., t_k.

The arguments of functions are explicit, except in two instances. The time argument on continuous functions of time is often omitted for convenience; e.g., $\underline{x}(t) = \underline{x}$. In the other instance, the sampling interval index (k) is used in place to time t_k.

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CHAPTER 2

A SURVEY OF SENSITIVITY ANALYSIS IN OPTIMAL CONTROL

Introduction and Background

The sensitivity of a control system was perhaps first mentioned by Bode [1] in his book published in 1945. His definition of the sensitivity of the system gain, to variation in a parameter, was

$$S^{-1} = \frac{dp}{dT}\,\frac{T}{p}, \tag{1}$$

where T is the system gain and p is a parameter of the system. For almost ten years after its introduction, there was very little written on sensitivity.

Beginning in 1955, work began to appear in which Bode's definition of sensitivity was inverted and related to other system characteristics. The principal contributors to this early work were Horowitz [2], Truxal [2,3] and Mason [4].

During the years 1957 through 1962, sensitivity received increased attention. The pole-zero sensitivity was studied [5,6,7], and sensitivity was related to root-locus properties [8]. The use of sensitivity analysis in linear system theory was presented by a number of people [9-15]. It was extended to sampled-data systems [16,17], and a

number of articles were written on the use of the sensitivity coefficients for system identification and adaptive control [18-22]. It was during this period that the need for a more general approach to sensitivity became more apparent. As a result, even more articles on the application of sensitivity analysis began to appear.

A number of important developments took place in 1963. Tomović published the first book devoted entirely to sensitivity analysis [23]. The book contains a formulation of theoretical and practical problems with the aim of encouraging more work on a general theory of sensitivity analysis. The close relationships, as well as the distinct differences between stability and sensitivity, are discussed.

Tomović considers a dynamic system with the mathematical model

$$F(\ddot{x},\dot{x},x,t,p) = 0, \tag{2}$$

where x is the state of the system and p is a single system parameter. (Note that the nomenclature used by Tomović has been changed for convenience.) The dynamic sensitivity coefficient is defined as the change in the state x due to variations in the parameter. This is expressed as

$$v(t,p) = \frac{dx(t,p)}{dp} . \tag{3}$$

The sensitivity equation is obtained by first taking the partial derivative of equation (2) with respect to p,

$$\frac{\partial F}{\partial \ddot{x}}\frac{\partial \ddot{x}}{\partial p} + \frac{\partial F}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial p} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial F}{\partial p} = 0, \tag{4}$$

and then substituting the relations

$$\frac{\partial \dot{x}}{\partial p} = \dot{v} \text{ and } \frac{\partial \ddot{x}}{\partial p} = \ddot{v}.$$

This yields the sensitivity equation

$$\frac{\partial F}{\partial \ddot{x}} \ddot{v} + \frac{\partial F}{\partial \dot{x}} \dot{v} + \frac{\partial F}{\partial x} v = - \frac{\partial F}{\partial p} . \tag{5}$$

The sensitivity equation is a linear differential equation and can be solved by analytical means. However, the book deals only with machine and experimental solution methods and gives several examples. One method is the simultaneous solution of the system, equation (2), and the sensitivity equation, equation (5), on an analog computer. This makes use of the connections between the two equations. Tomović also notes that the structural similarity of the two equations facilitates solution by a digital computer.

The book also includes a discussion on the solution of the sensitivity equations by simulation. This method, based on earlier work by Bihovski [24], assumes that the dynamic system has been realized and the sensitivity coefficients are to be obtained by direct measurement. Bihovski's work is also the basis for the structural method of obtaining the sensitivity coefficients of linear systems.

The structural method has been presented in a number of places [25] and its primary advantage is that the sensitivity coefficients for a number of parameter variations are obtained simultaneously. Another feature is the

simplicity of the analog model. The only portion of the system that must be simulated in detail is the portion that is to be studied closely.

The sensitivity coefficient (as a function of time) about a particular parameter p_o is useful, but a knowledge of its values in parameter space is even more useful. Such problems as structural sensitivity and the effect of the variations of a number of parameters can be studied by means of the sensitivity coefficients in parameter space.

The problem of inverse sensitivity is also discussed in the book. In this formulation, the variations in x are known, and it is desired to determine the corresponding variations in the parameter p. The book also includes brief discussions on adaptive control of invariant systems and performance adjustment of dynamic systems.

In 1963, another book written by Horowitz [26] gave considerable space to the subject of sensitivity. Horowitz used sensitivity in presenting the design and synthesis of linear multivariable continuous and sampled-data systems. Another indication of the growing importance of sensitivity was a session on it at a circuit and system theory conference [27]. It was also in 1963 that sensitivity was first applied to optimal control systems [28,29]. The subject of sensitivity in optimal control has received a great deal of attention since its introduction and its development will be discussed in the next section.

To see the development for a system more general than that of equation (2), consider the system

$$\dot{\underline{x}} = \underline{f}\{\underline{x}(t), \underline{p}\} \qquad \underline{x}(0) = \underline{x}_o, \tag{6}$$

where $\underline{x}$ is an n dimensional state vector and $\underline{p}$ is an m dimensional parameter vector. A sensitivity equation and sensitivity coefficients for parameters other than those that change system order and initial conditions can be derived [25]. For small changes in $\underline{p}$, a first-order approximation for the corresponding change in $\underline{x}$ is

$$\underline{\Delta x} = \sum_{j=1}^{m} \underline{v}_j \Delta p_j + \cdots, \tag{7}$$

where $\underline{v}_j$ is the sensitivity vector $\underline{v}_j = \left[\frac{\partial x_1}{\partial p_j} \cdots \frac{\partial x_n}{\partial p_j}\right]^T$. The i^{th} component of $\underline{v}_j$ is the sensitivity coefficient

$$v_{ij} = \frac{\partial x_i}{\partial p_j}\Bigg|_{\substack{\Delta p_1 = 0, \\ \vdots \\ \Delta p_m = 0}} \tag{8}$$

which represents the variation in the i^{th} component of the state vector $\underline{x}$ due to a change in the j^{th} component of the parameter vector $\underline{p}$. The sensitivity equation is

$$v_{kj} = \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} v_{ij} + \frac{\partial f_k}{\partial p_j}, \tag{9}$$

where $k = 1, \ldots, n$, and $j = 1, \ldots, m$, and the initial conditions are $v_{kj}(0) = 0$.

If the sensitivity to changes in initial conditions, $\underline{x}(0)$, of equation (6) is desired, the initial conditions for equation (9) must be $v_{kj}(0) = 1$ [25]. In order to handle parameters that change the order of the system, the system equations

$$\dot{x}_i = f_i(x_1,\ldots,x_n;x_{n+1},\ldots,x_{n+r};t) \qquad (i = 1,\ldots,n)$$

$$\lambda\dot{x}_{n+s} = f_{n+s}(x_1,\ldots,x_n;x_{n+1},\ldots,x_{n+r};t) \quad (s = 1,\ldots,r), \tag{10}$$

are used. Note that λ changes the order of the system and when $\lambda = 0$, equation (10) represents the original system. The sensitivity equations are

$$v_k = \sum_{i=1}^{n+r} \frac{\partial f_k}{\partial x_i} v_i \quad (k = 1,\ldots,n)$$

$$v_{n+s} = \sum_{i=1}^{n+r} \frac{\partial f_{n+s}}{\partial x_i} v_i - \frac{dx_{n+s}}{dt} \quad (s = 1,\ldots,r) \tag{11}$$

with $v_h(0) = 0$, where $(h = 1,\ldots,n+r)$ [25]. Several other papers on parameters that change the order of the system have been published [30,31].

It should be noted that the sensitivity functions can be determined by other methods. For example, the method of undetermined coefficients, difference equations, and asymptotic expansions can be used and do not require equation (6) to be regular in p [32]. They also allow a much wider class of perturbations to be treated. Some of the other types of perturbations that have recently been

incorporated into the framework of sensitivity are: frequency of oscillation, time delay, sampling rate, integration step, and "amount" of nonlinearity [32].

The rapid progress has been made possible, in part, by drawing from other areas such as network theory, error analysis in analog computers, and the theory of differential equations. These recent developments in sensitivity analysis have been covered by several articles [25,32], and an international symposium on sensitivity analysis [33]. Also, sensitivity was the subject of several sessions at a conference on circuit and system theory [34].

Sensitivity In Optimal Control

Since 1963, when sensitivity was first applied to optimal control, a number of uses for sensitivity in optimal control have been presented. The sensitivity of such quantities as the performance index, the state vector, and the terminal state have been studied for variations in the plant parameters, the state vector, and the control vector. The following section will present some of the results of the work in this area.

Dorato [28] used sensitivity to determine the variation in the performance index

$$J = \int_{t_o}^{T} F(\underline{x},\underline{u})dt, \tag{12}$$

due to changes in the plant parameters. The plant state

vector $\underline{x}(t)$ is related to the plant control vector $\underline{u}(t)$ by the vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}\{\underline{x}(t), \underline{u}(t), \underline{p}\}, \tag{13}$$

where $\underline{p}$ represents the set of plant parameters. For this case, the optimal closed-loop control law is of the form

$$\underline{u}(t) = Q\{\underline{x}(t), \underline{p}; t\}. \tag{14}$$

For changes in $\underline{p}$ from the nominal $\underline{p}_o$, the change in the performance index is

$$\Delta J = J(\underline{p}_o) - J(\underline{p}).$$

Dorato considers small variations in $\underline{p}$ and writes

$$\Delta J \cong dJ = \frac{\partial J}{\partial p_1} dp_1 + \ldots + \frac{\partial J}{\partial p_m} dp_m, \tag{15}$$

or

$$\Delta J \cong \frac{\partial J}{\partial \underline{p}} d\underline{p}, \tag{16}$$

where $\frac{\partial J}{\partial \underline{p}}$ is the performance index sensitivity vector. This can be written as

$$\frac{\partial J}{\partial \underline{p}} = \int_{t_o}^{T} \frac{\partial F}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{p}} dt, \tag{17}$$

where $\frac{\partial \underline{x}}{\partial \underline{p}}$ is a matrix and is the solution to the sensitivity equation [35]. Using this notation, equation (9) can be rewritten as

$$\frac{d}{dt}\left[\frac{\partial \underline{x}}{\partial \underline{p}}\right] = \frac{\partial \underline{f}}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{p}} + \frac{\partial \underline{f}}{\partial \underline{p}}. \tag{18}$$

Dorato suggested that sensitivity might be a useful criterion to use in comparing open-loop and closed-loop control. Cruz and Perkins have investigated this problem and the more general problem of the sensitivity of multivariable systems [36-40]. They introduce a sensitivity matrix $S(s)$ which relates $E_o(s)$ to $E_c(s)$ by the relationship

$$E_c(s) = S(s)E_o(s), \tag{19}$$

where $E_c(s)$ is the output error due to plant parameter variations of a closed-loop realization of the system. The output error of the open-loop system is represented by $E_o(s)$. Note that in order to have a meaningful comparison, the output of both the open-loop and the closed-loop systems must be equal if there are no plant parameter variations.

Cruz and Perkins have related the sensitivity matrix $S(s)$ to a matrix generalization of return difference for multivariable, linear, time-invariant feedback systems. Also, for single-input, single-output systems, the sensitivity matrix is compatible with the classical definition of sensitivity. However, in the application of sensitivity to optimal control systems, one of the most useful aspects of their work has been the idea of a "comparative" sensitivity.

Another definition of sensitivity that is of a "comparative" nature was introduced by Rohrer and Sobral [41].

In order to avoid having to completely specify the conditions associated with the normal or "absolute" definitions of sensitivity, they define a relative sensitivity

$$S^R[\underline{u}(t),\underline{p}] = \frac{J[\underline{u}(t),\underline{p}] - J[\underline{u}^o(t),\underline{p}]}{|J[\underline{u}^o(t),\underline{p}]|}, \tag{20}$$

where $J[\underline{u}^o(t),\underline{p}]$ represents the performance index associated with the system when driven by the optimum control, $\underline{u}^o(t)$, for the given set of plant parameters $\underline{p}$. $J[\underline{u}(t),\underline{p}]$ is the performance index when the control, $\underline{u}(t)$, is not the optimum for the given set of parameters $\underline{p}$. They use the calculus of variations to show that,

$$S^R[\underline{u}(t),\underline{p}] \cong \frac{\delta J[u^o(t),u(t),p]}{|J[\underline{u}^o(t),\underline{p}]|} \tag{21}$$

for $\underline{u}^o(t)$ interior to its allowable set U, and

$$S^R[\underline{u}(t),\underline{p}] \cong \frac{\delta^2 J[u^o(t),u(t),p]}{|J[\underline{u}^o(t),\underline{p}]|} \tag{22}$$

when $\underline{u}^o(t)$ is on the boundary of U. Note that relative sensitivity approaches zero as a system approaches its optimal performance. Also, it should be pointed out that the relative sensitivity is a function of both the control $\underline{u}(t)$ and the parameters $\underline{p}$.

Rohrer and Sobral use relative sensitivity to define a plant sensitivity

$$S^M[\underline{u}(t)] = \underset{\underline{p}\in P}{\mathrm{Max}}\{S^R[\underline{u}(t),\underline{p}]\}, \tag{23}$$

which is the maximum value of relative sensitivity for all

parameters of the allowable set P. This definition of plant sensitivity could be used as a design criterion and the optimization would seek the control $\underline{u}^*(t)$ which minimizes the plant sensitivity $S^M[\underline{u}(t)]$. Thus, if the plant parameters were known to vary over a certain range, the control $\underline{u}^*(t)$ would minimize the maximum deviation from optimal performance for parameter variations over the specified range.

Rohrer and Sobral have also defined a plant sensitivity that is useful in systems in which the plant parameters are given as random variables. This definition is based on the expected value of the relative sensitivity and can be stated as

$$S^E[\underline{u}(t)] = \underset{\underline{p}\in P}{E}\{S^R[\underline{u}(t),\underline{p}]\}, \tag{24}$$

where "E" indicates the expected value. Here the optimization would seek the control $\underline{u}^*(t)$ which minimizes $S^E[\underline{u}(t)]$. This would minimize the average or expected deviation from the optimal performance.

A similar optimization procedure using a game theory approach has been formulated [42,43,44]. This method is useful since both large and small plant parameter variations can be considered and the controller structure need not be fixed [42]. Dorato and Kestenbaum [43] consider a fixed controller structure with a controller parameter p_c. The plant dynamics are given by

$$\dot{\underline{x}}(t) = \underline{f}\{\underline{x}(t),\underline{u}(t),p_p\}, \tag{25}$$

where p_p is the plant parameter. If $p_c = p_p$, then the controller generates the control

$$\underline{u}(t) = Q\{\underline{x}(t), p_c; t\}, \tag{26}$$

which is optimal and the performance index

$$J = \int_{t_o}^{T} F(\underline{x},\underline{u})\,dt, \tag{27}$$

is minimized. In the problem they formulate, all that is known about p_p is that it lies somewhere in the range $p_1 \leq p_p \leq p_2$, and, therefore, the performance index is a function of p_p and p_c. The object of the optimization is to determine the "best" value of p_c.

Since p_p is known only to range from p_1 to p_2, the desirable controller parameter p_c^o should keep the performance index equal to or less than some value, J^o, for all values of p_p in its expected range. This can be expressed as

$$J(p_c^o, p_p) \leq J^o, \text{ for } p_1 \leq p_p \leq p_2. \tag{28}$$

Also, the inequality

$$J^o \leq J(p_c, p_p^o), \text{ for } p_1 \leq p_c \leq p_2, \tag{29}$$

must also hold if J^o is to be as low as possible.

In the game theory interpretation, $J(p_c, p_p)$ is the value of the game or the "pay-off function" and the players or "antagonists" are p_p and p_c. The pair (p_c^o, p_p^o) is an optimal or pure strategy and the type of

game is infinite or continuous. The conditions for optimal strategies to be pure are [43],

$$\min_{p_c} \max_{p_p} J(p_c, p_p) = \max_{p_p} \min_{p_c} J(p_c, p_p) \tag{30}$$

and the existence of numbers p_c^o, p_p^o, and J^o such that

$$J(p_c^o, p_p) \leq J^o \leq J(p_c, p_p^o) \tag{31}$$

Recently, Pagurek [45] has presented some interesting results for linear systems. He formulates the sensitivity of the performance index into the structure of the Hamilton-Jacobi equation and shows that the open- and closed-loop performance index sensitivity functions are the same. This approach is useful in that sensitivity analysis can be carried out by the same technique used to obtain the optimal control law. His work has been extended to the nonlinear case by Witsenhausen [46].

Šiljak and Dorf [47] point out that most applications of sensitivity to optimal control do not use sensitivity as a criterion for determining the optimal control, but determine sensitivity after the optimal control has been synthesized. In order to avoid this, they use the time-domain sensitivity technique [19,48] and introduce a general index of optimality which includes both sensitivity and performance characteristics. Thus, the optimal control synthesized satisfies sensitivity and optimality requirements simultaneously.

In order to include sensitivity in a general index, the usual index of performance

$$J = \int_{t_o}^{T} F(\underline{x},\underline{u}t)dt, \tag{32}$$

is altered to also include the sensitivity functions. The resulting generalized index of optimality is

$$J = \int_{t_o}^{T} G(\underline{x},\underline{u},\underline{v}_1,\underline{v}_2,\ldots,\underline{v}_m,t)dt, \tag{33}$$

where the sensitivity functions are $\underline{v}_1 = \frac{\partial x}{\partial p_i}$ and p_i is the i^{th} variable parameter. The sensitivity functions, $\underline{v}_i$, should appear in the index as squares or magnitudes to avoid the canceling effects of a change in sign. Also, the authors indicate the usefulness of a weighting function which would allow certain sensitivity functions to receive different emphasis. The resulting control law $\underline{u}^o(t)$ optimizes both sensitivity and performance.

In addition to the sensitivity of the performance index, the sensitivity of the terminal state of an optimal control system has been studied. Gavrilović, Petrović, and Šiljak [29] investigated it by the adjoint method in one of the first articles applying sensitivity to optimal control. More recently, Holtzman and Horing [49] used variational techniques to study the sensitivity of the terminal conditions of both open- and closed-loop optimal systems. An important part of their work is the inclusion of sensitivity prior to optimization for the open-loop

system. This allows the sensitivity of terminal conditions to be prespecified or constrained. The results of their work confirm that the closed-loop configuration has superior sensitivity characteristics.

The sensitivity to variations in the plant parameters has been the object of most of the work on sensitivity in optimal control. However, Bélanger [50] has investigated the effects of variations in the control. He points out that this is useful in suboptimal control and also in studies of the sensitivity of the computation of the desired control. Control variations have also been studied by Gavrilović, Petrović, and Šiljak [29]. They consider variations in control by changes in the initial conditions of the adjoint system instead of letting the control vary directly.

Bélanger considers both "weak" and "strong" variations in the control. For the weak or "continual" variation, the actual control differs from the desired control by an infinitesimal amount $\epsilon\eta(t)$. Tolerances on the control can be set by limiting $\eta(t)$. The strong or "intermittent" variation causes actual control to differ from the desired control by large amounts, but only during infinitesimal intervals of time. The continual variation is applicable when the control is continuous and the intermittent variation is useful for "bang-bang" control.

There are two effects of a variation in control. One result of a variation in the control would be the failure

to hit a desired target or terminal state. The other effect would be variations in the value of the performance index. Bélanger has considered both effects for the case in which the actual target has been replaced by an ideal target. This is useful since control tolerances necessary to hit the actual target can be determined and variation in cost calculated. For example, if the target were a small sphere, the control to hit a point at its center would be calculated and then tolerances determined for this control to insure that the sphere will always be reached.

Summary

The first portion of this survey presents a historical development of sensitivity analysis in automatic control. The various sensitivity functions, vectors, and coefficients are defined and several methods for their calculation are discussed. Also, some of the recent contributions toward a generalized approach to sensitivity analysis are presented. With this background information, the application of sensitivity to optimal control is discussed next.

A variety of uses of sensitivity in optimal control has been formulated in recent years. The discussion includes the sensitivity of the performance index, the state vector and the terminal state for variations in plant parameters, controller parameters, the state vector, and the control vector. Also included are some of the

optimization schemes which make use of sensitivity to optimize system performance. The use of sensitivity in establishing the control tolerance necessary for target states is discussed.

In looking over the variety of methods and definitions that has been used for the sensitivity analysis of optimal systems, it is obvious that no single method is completely satisfactory. The subject is still in the early stages of its development and there is much need of a general, comprehensive approach. There is a great deal of interest in this subject and no doubt considerable progress will be made in the near future.

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CHAPTER 3

SENSITIVITY ANALYSIS FOR DISCRETE SYSTEMS

Introduction

The utility of sensitivity analysis in discrete systems has been recognized [1] and a few papers have been presented [2,3,4]. The early work on sensitivity was influenced by the analog computer, and as a result, most methods developed to determine sensitivity "factors" are in the form of solutions of continuous equations.

In view of the wide spread use of discrete and hybrid systems, a discrete approach to sensitivity analysis would be useful. It would certainly be beneficial for the sensitivity of discrete systems and might also be useful in analysis of continuous systems.

The existing continuous sensitivity equations can be solved on a digital computer. However, the discrete version should be more convenient since an increasing amount of system simulation is being done on digital computers.

In developing a discrete approach to sensitivity, discrete versions of the existing continuous factors and equations will be formulated. Also, factors that are only useful for discrete systems will be investigated.

Parameter Sensitivity of Discrete Systems

One of the most useful sensitivity factors is the sensitivity of the state variable to changes in system parameters. In the following discussion, state-parameter sensitivity is developed by two methods. The change in state is first expressed in terms of a perturbation matrix and then developed by means of a sensitivity-vector function.

The most general representation of a lumped discrete system is given by the equation

$$\underline{x}(k) = \underline{f}\{\underline{x}(k-1),\underline{u}(k-1),\underline{p},t_{k-1},t_k\}, \tag{1}$$

where $\underline{x}(k)$ is a n-dimensional state vector, $\underline{u}(k)$ is a m-dimensional control vector, and $\underline{p}$ is a constant r-dimensional parameter vector. The sampling interval T_k is the time between two consecutive sampling instants,

$$T_k = t_{k+1} - t_k. \tag{2}$$

For convenience, the arguments of $\underline{x}$ and $\underline{u}$ will not include the time t explicitly. Instead, time will be designated by the particular sampling interval. For example; $\underline{x}(t_k) = \underline{x}(k)$.

If the discrete system is linear, it can be represented by the equation

$$\underline{x}(k) = A(k-1)\underline{x}(k-1) + B(k-1)\underline{u}(k-1), \tag{3}$$

where $A(k-1)$ and $B(k-1)$ are $A(t_{k-1},t_k)$ and $B(t_{k-1},t_k)$ during the sampling interval $T_{k-1} = (t_k - t_{k-1})$.

For linear systems, this is often more convenient than the form given in equation (1). If the system of equation (3) is also stationary, the matrices A and B are constant;

$$A(j) = A(i) = A \text{ and } B(j) = B(i) = B.$$

In the following developments, some of the methods are only applicable for linear systems, and equation (3) will be used.

Perturbation matrix approach

One method for determining the variation in the state $\underline{x}(k)$ for changes in system parameters makes use of a perturbation matrix. This method is applicable only to linear systems and will be developed from equation (3) and its solution, which is [5],

$$\underline{x}(k) = \Phi(k,0)\underline{x}(0) + \sum_{j=0}^{k-1} \Phi(k,j+1)B(j)\underline{u}(j), \tag{4}$$

where the transition matrix is given by,

$$\Phi(k,j) = \prod_{i=j}^{k-1} A(i) = A(k-1)A(k-2)\cdots A(j+1)A(j). \tag{5}$$

If the system is stationary, the transition matrix is

$$\Phi(k) = A^{k} = A\cdot A\cdot A\cdots; \tag{6}$$

and the solution vector can be written as

$$\underline{x}(k) = \Phi(k)\underline{x}(0) + \sum_{j=0}^{k-1} \Phi(j)B\underline{u}(k-j-1). \tag{7}$$

The derivation will be carried out for a stationary system. The results will also be given for time-varying systems. To determine the effects of a change in parameters, let the matrix A of a stationary system change by an amount ϵC. Equation (3) can be written for the perturbed system [6]

$$\underline{\tilde{x}}(k) = [A + \epsilon C]\underline{\tilde{x}}(k-1) + B\underline{u}(k-1). \tag{8}$$

If equation (8) is written for successive values of k, and certain substitutions made, the system transition equation can be derived for $\underline{\tilde{x}}(k)$ in terms of the control over the interval, the initial value $\underline{x}(0)$, and the system parameters. The resulting equation is

$$\underline{\tilde{x}}(k) = \Phi(k)\underline{x}(0) + \sum_{j=0}^{k-1} \Phi(j)B\underline{u}(k-j-1)$$

$$+ \epsilon\sum_{j=0}^{k-1} \Phi(k-j-1)C\Phi(j)\underline{x}(0)$$

$$+ \epsilon\sum_{i=0}^{k-2} \sum_{j=0}^{k-i-2} \Phi(k-j-i-2)C\Phi(j)B\underline{u}(i) + O(\epsilon^2). \tag{9}$$

The change in $\underline{x}(k)$, $\underline{\Delta x}(k) = \underline{\tilde{x}}(k) - \underline{x}(k)$, can be determined by subtracting equation (7) from equation (9). The first-order change in $\underline{x}(k)$ is obtained by neglecting the higher order terms. For a stationary linear system,

the first-order change is

$$\underline{\Delta x}(k) = \epsilon \sum_{j=0}^{k-1} \Phi(k-j-1)C\ (j)\underline{x}(0)$$

$$+ \epsilon \sum_{i=0}^{k-2} \sum_{j=0}^{k-i-2} \Phi(k-j-i-2)C\Phi(j)B\underline{u}(i). \tag{10}$$

A similar equation gives the first-order change in $\underline{x}(k)$ for a time-varying linear system.

$$\begin{aligned}\underline{\Delta x}(k) &= \epsilon \sum_{j=0}^{k-1} \Phi(k,j+1)C(j)\Phi(j,0)\underline{x}(0) \\ &+ \epsilon \sum_{i=0}^{k-2} \sum_{j=0}^{k-i-2} \Phi(k,j+i+2)C(j+i+1)\Phi(j+i+1,i+1)B(i)\underline{u}(i).\end{aligned} \tag{11}$$

The first-order effects of ϵC can also be derived in the form of a difference equation

$$\underline{\Delta x}(k) = A(k-1)\ \underline{\Delta x}(k-1) + \epsilon C(k-1)\underline{x}(k-1), \tag{12}$$

with $\underline{\Delta x}(0) = 0$. Thus, the first-order change at each sampling instant can be calculated from the state and the first-order error at the preceding sampling instant.

An exact difference equation for the change in a state $\underline{x}(k)$ due to the perturbations $\epsilon C(k)$ is

$$\underline{\Delta x}(k) = [A(k-1) + \epsilon C(k-1)]\underline{\Delta x}(k-1) + \epsilon C(k-1)\underline{x}(k-1), \tag{13}$$

where $\underline{\Delta x}(0) = 0$. Note that the exact change given by equation (13) differs by only one term from the first-order change of equation (12). An equation of the form of equation (11) could be derived for the exact change, but it is too complex to be useful in computations.

Equation (12) is not so accurate as equation (13), but it is more general. If equation (12) is divided by ϵ, the resulting difference equation is

$$\frac{\underline{\Delta x}(k)}{\epsilon} = A(k-1)\frac{\underline{\Delta x}(k-1)}{\epsilon} + c(k-1)\underline{x}(k-1). \tag{14}$$

The vector $\frac{\underline{\Delta x}(k)}{\epsilon}$ can be calculated along with $\underline{x}(k)$ and then the first-order error for any small ϵ can be determined from $\frac{\underline{\Delta x}(k)}{\epsilon}$. This generalization cannot be made on equation (13).

The equations derived in this section are not sensitivity equations by definition. However, they do give the change in the state due to known perturbations of system parameters. Note that by carefully selecting the elements of the matrix C, any number of the system parameters of A can be varied. Equations (10) and (11) give the first-order change in any state in terms of $\underline{x}(0)$ and the control over the interval. Equations (12) and (13) give the first-order and exact change in any state in terms of the preceding state and change. These equations will be illustrated in an example problem and then will be used to check the accuracy of other sensitivity methods.

Sensitivity vector-function approach

Another approach to the sensitivity of discrete systems is very similar to the sensitivity vectors and sensitivity functions of continuous systems [7]. This method does not require the system to be stationary or linear. A sensitivity vector can be defined as

$$\underline{v}_i(k) = \frac{\partial \underline{x}(k)}{\partial p_i} = \left[\frac{\partial x_1(k)}{\partial p_i} \quad \frac{\partial x_2(k)}{\partial p_i}, \ldots, \frac{\partial x_n(k)}{\partial p_i}\right]^T, \tag{15}$$

where the sensitivity functions are the components,

$$v_{ij}(k) = \left.\frac{\partial x_j(k)}{\partial p_i}\right|_{\substack{\Delta p_i = 0 \\ \vdots \\ \Delta p_r = 0}} . \tag{16}$$

There will be a sensitivity vector for each of the r parameters of the parameter vector $\underline{p}$. Using the sensitivity vectors, a first-order approximation of the change in $\underline{x}(k)$ can be written as

$$\underline{\Delta x}(k) = \sum_{j=1}^{r} \underline{v}_j(k) p_j . \tag{17}$$

The method used to calculate the sensitivity coefficients is of primary importance. In continuous analysis, a differential equation is formulated and its solution yields the sensitivity functions. For the discrete approach, a difference equation would be desirable.

To formulate a sensitivity difference equation, take the partial derivative of equation (1) with respect to p_i. This yields the equation

$$\frac{\partial \underline{x}(k)}{\partial p_i} = \frac{\partial \underline{f}(k-1)}{\partial \underline{x}(k-1)} \cdot \frac{\partial \underline{x}(k-1)}{\partial p_i} + \frac{\partial \underline{f}(k-1)}{\partial p_i}. \tag{18}$$

Note that the terms involving $\underline{u}$ are not present, as it is assumed that the control is not dependent upon the changing parameters. Substituting equation (15) into equation (18) results in a difference equation for the sensitivity vectors of a discrete system,

$$\underline{v}_i(k) = \frac{\partial \underline{f}(k-1)}{\partial \underline{x}(k-1)} \underline{v}_i(k-1) + \frac{\partial \underline{f}(k-1)}{\partial p_i}, \tag{19}$$

where $\underline{v}_i(0) = 0$. The sensitivity functions are determined from an equation for the components of equation (19),

$$v_{ij}(k) = \sum_{s=1}^{n} \frac{\partial f_j(k-1)}{\partial x_s(k-1)} v_{is}(k-1) + \frac{\partial f_j(k-1)}{\partial p_i}, \tag{20}$$

where $v_{ij}(0) = 0$.

The sensitivity vector function approach of equations (19) and (20) is in general more useful than the transition matrix approach discussed in the previous section. The primary advantage of the sensitivity vector function approach is that the varying parameters are not restricted to the elements of A. Both methods are illustrated by an example problem in a later section.

Sampling Interval Sensitivity of Sampled Systems

The performance of a sampled-data system is usually very sensitive to changes in the sampling period. Therefore, knowledge of the sampling interval sensitivity should be quite useful. Sampling interval sensitivity could be used as a basis for selecting the universal sampling interval or for selecting individual sampling intervals. These two applications are significant in that they illustrate two distinct sampling interval sensitivity functions. In selecting a universal sampling interval, the fixed sampling rate that optimizes system performance can be determined with the aid of a "global" sensitivity function. Individual sampling intervals would be selected by means of a "local" sensitivity function [3]. In the following work, both "global" and "local" sampling interval sensitivity functions are defined, and the equations necessary for their calculation will be presented.

Global sampling interval sensitivity

If a system has a fixed sampling interval, then $T_i = T_j$ in equations (1) and (3), and a global sampling interval sensitivity function can be defined as

$$\underline{v}_T(k) = \lim_{\Delta T \to 0} \frac{\underline{x}\{k(T+\Delta T)\} - \underline{x}\{k(T)\}}{\Delta T}. \tag{21}$$

Equation (21) is an expression for the change in the state at the k^{th} sampling instant if every sampling interval is changed by ΔT. Note that if K samples are required to cover the interval of interest, then the size of ΔT must obey the inequality

$$\Delta T < \frac{\text{Total Time}}{K} . \tag{22}$$

This is necessary since the i^{th} sampling instant is shifted by $i \cdot \Delta T$ and the maximum shift $K \cdot \Delta T$ must not move the k^{th} sampling instant into another sampling interval [3].

If the state x(k) is a continuous function of T, the global sampling interval sensitivity function of equation (21) can be written as

$$\underline{v}_T(k) = \frac{\partial \underline{x}(k)}{\partial T} . \tag{23}$$

A discrete sensitivity equation for $\underline{v}_T(k)$ can be derived by taking the partial derivative of equation (1) with respect to T. If parameter variations are not considered,

$$\frac{\partial \underline{x}(k)}{\partial T} = \frac{\partial \underline{f}(k-1)}{\partial \underline{x}(k-1)} \frac{\partial \underline{x}(k-1)}{\partial T} + \frac{\partial \underline{f}(k-1)}{\partial \underline{u}(k-1)} \frac{\partial \underline{u}(k-1)}{\partial T} + \frac{\partial \underline{f}(k-1)}{\partial T} . \tag{24}$$

Substituting with equation (23) yields the discrete sensitivity equation

$$\underline{v}_T(k) = \frac{\partial \underline{f}(k-1)}{\partial \underline{x}(k-1)} \underline{v}_T(k-1) + \frac{\partial \underline{f}(k-1)}{\partial \underline{u}(k-1)} \frac{\partial \underline{u}(k-1)}{\partial T} + \frac{\partial \underline{f}(k-1)}{\partial T} , \tag{25}$$

subject to the initial conditions $\underline{v}_T(0) = 0$.

If system is linear, then equation (3) is used to represent the system, and the sensitivity equation is

$$\underline{v}_T(k) = A(k-1)\underline{v}_T(k-1) + \frac{\partial A(k-1)}{\partial T}\underline{x}(k-1) + \frac{\partial B(k-1)}{\partial T}\underline{u}(k-1) + B(k-1)\frac{\partial \underline{u}(k-1)}{\partial T}, \tag{26}$$

with the initial conditions $\underline{v}_T(0) = 0$.

The n-dimensional global sensitivity vector will have a set of values for each sampling interval. The i^{th} component of $\underline{v}_T(k)$ indicates the effect of changes in every sampling interval on the i^{th} component of the state vector $\underline{x}(k)$ at the k^{th} sampling instant. Therefore, it can be used to extrapolate solutions about a nominal sampling interval T_o. For small values of k, the first two terms of a Taylor's series,

$$\underline{x}[k(T_o + \Delta T)] \cong \underline{x}(kT) + \underline{v}_T(k)\Delta T, \tag{27}$$

will give good results. The accuracy of the approximation deteriorates as k increases. This is characteristic of sensitivity methods that involve frequency [3]. Global sampling interval sensitivity will be illustrated in an example problem.

Local sampling interval sensitivity

If the sampling interval is not held constant, then

the global sensitivity defined in the preceding section does not apply. Therefore, a local sensitivity function is required to determine the effect of a change in the k^{th} sampling period on the state at the $(k+1)^{st}$ sampling instant. A local sampling interval sensitivity function can be defined as

$$\underline{v}(k) = \lim_{\Delta t \to 0} \frac{\underline{x}(t_k+\Delta t) - \underline{x}(t_k)}{\Delta t} = \frac{\partial \underline{x}(t_k)}{\partial t_k}. \tag{28}$$

To derive a sensitivity equation for local sampling interval sensitivity, the system equation is first written in the form

$$\underline{x}(k) = \underline{f}\{\underline{x}(k-1), \underline{u}(k-1), t_{k-1}, t_k\}. \tag{29}$$

To determine $\underline{v}(k)$, differentiate equation (29) with respect to t_k,

$$\underline{v}(k) = \frac{\partial \underline{f}\{\underline{x}(k-1), \underline{u}(k-1), t_{k-1}, t_k\}}{\partial t_k}. \tag{30}$$

If the system is linear, the sensitivity equation is

$$\underline{v}(k) = \frac{\partial A(t_{k-1}, t_k)}{\partial t_k} \underline{x}(k-1) + \frac{\partial B(t_{k-1}, t_k)}{\partial t_k} \underline{u}(k-1). \tag{31}$$

Note that the terms $A(t_{k-1},t_k)\frac{\partial \underline{x}(k-1)}{\partial t_k}$ and $B(t_{k-1},t_k)\frac{\partial \underline{u}(k-1)}{\partial t_k}$ are missing from equation (31) because both $\frac{\partial \underline{x}(k-1)}{\partial t_k}$ and $\frac{\partial \underline{u}(k-1)}{\partial t_k}$ are zero. This is true since $\underline{x}(k-1)$ and $\underline{u}(k-1)$

are insensitive to changes in the following k^{th} sampling instant.

Equation (31) gives the sensitivity of $\underline{x}(k)$ to changes in T_{k-1} only. Similar to global sensitivity, local sensitivity can be used for extrapolation, but only about t_k. Also, additional insight into the relationship between local and global sensitivity is gained by noting that for a linear system, the global sensitivity is the sum of all preceding local effects [3].

Example Problems

The purpose of this section is to illustrate the calculation of the various sensitivity functions that have been defined.

Problem 1

As a first example, consider the system of figure 3-1. The equation for the continuous system is

$$\dot{x}(t) = -ax(t) + u(t). \tag{32}$$

The discrete version of the system is

$$x(k) = Ax(k-1) + Bu(k-1), \tag{33}$$

where

$$A = \exp\{-a(t_k - t_{k-1})\}, \tag{34}$$

and

$$B = \int_{t_{k-1}}^{t_k} \exp\{-a(t_k-\eta)\}\,d\eta. \tag{35}$$

For a sampling interval of 0.1 seconds and a = 1.0, the coefficients are: A = 0.90484 and B = 0.09516. The system was simulated on a digital computer and then response for a unit step is shown in figure 3-2.

To illustrate the perturbation matrix approach for parameters variation, equation (14) for the generalized first-order change, was programmed on a digital computer. The solution, $\frac{\Delta x(k)}{\epsilon}$, for a unit step input is plotted in figure 3-3. It should be noted that the response is not continuous as shown in figure 3-3, but is actually a series of discrete data points.

The sensitivity vector approach given in equation (19) was also used. For variations in A, this first-order system will have only one sensitivity vector with one component. The solution to equation (19) is identical to that of equation (14). Thus, the sensitivity vector and the perturbation matrix methods have the same solution. Figure 3-3 represents the sensitivity of state x to variations in the parameter A when determined by either method.

To check the accuracy of both methods, the system was simulated with A = 0.89584, B = 0.09516 and T = 0.1 seconds. This is equivalent to perturbing A by -0.009.

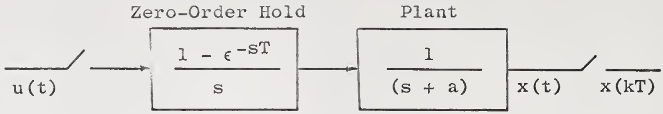

Figure 3-1 First-Order System

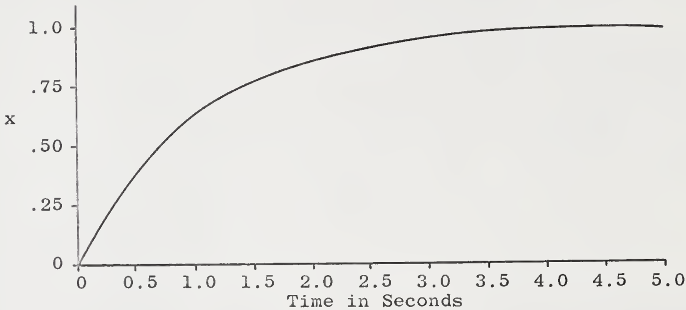

Figure 3-2 Unit Step Response

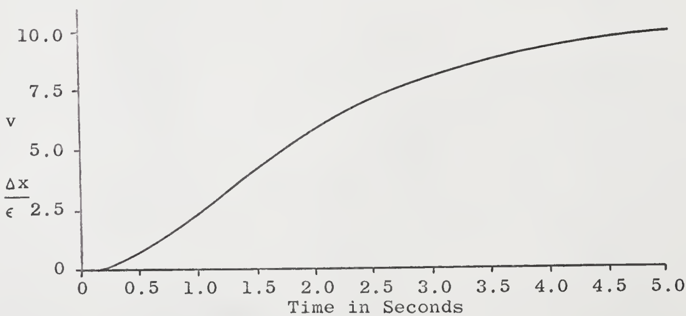

Figure 3-3 Generalized First-Order
Difference and Sensitivity

Figure 3-4 contains a plot of the difference between the original response x and the perturbed response x. For comparison, the figure also includes a curve for the first-order change calculated by equation (12) with $\epsilon C = -0.009$. The exact change, given by equation (13) with $\epsilon C = -0.009$, is identical to the difference in x from the two simulations. The first-order change was calculated by equation (10) for several values of k and the results agree with those of equation (12).

The accuracy of the sensitivity vector approach was checked by using equation (17) with $p = -0.009$. The calculated first-order change in x is identical to that of the perturbation matrix approach. Thus, the first-order error curve of figure 3-4 represents both methods.

The global sampling interval sensitivity function of equation (26) is shown in figure 3-5 for a unit step input and $T = 0.1$ seconds. The coefficients are; $A = 0.90484$ and $B = 0.09516$. Figure 3-5 also contains a plot of the local sampling interval sensitivity function as given in equation (31). The extrapolation by equation (27) of $\Delta x(k)$ for several values of T_o and ΔT, was compared with the actual change in x(k) when the system was simulated with $T = T_o + \Delta T$. The results were excellent as long as inequality (22) held. It should be pointed out that the $\Delta x(k)$ predicted by global or local sensitivity is not an error, but the change in x(k) due to a variation in T or T_k.

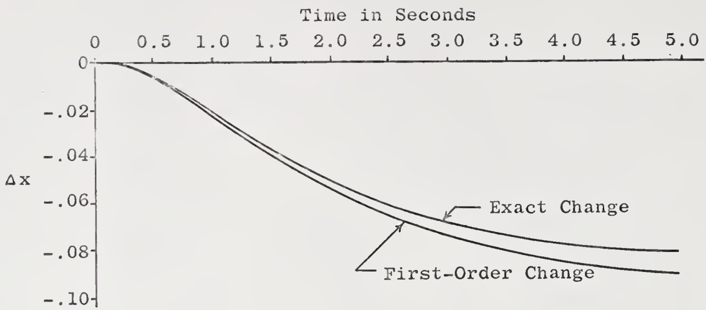

Figure 3-4 Exact and First-Order Change

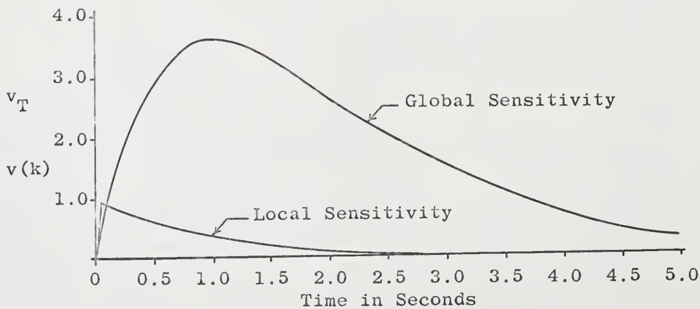

Figure 3-5 Global and Local Sensitivity

Problem 2

The sensitivity functions have also been calculated for the system in figure 3-6. The discrete version of the system is

$$\underline{x}(k) = A\underline{x}(k-1) + \underline{b}u(k-1). \tag{36}$$

The coefficient matrix A has the components:

$$a_{11} = \exp(-3\eta)[\cos(4\eta) + .75\sin(4\eta)] \tag{37}$$

$$a_{12} = (25/24)\exp(-3\eta)\sin(4\eta) \tag{38}$$

$$a_{21} = -1.5\exp(-3\eta)\sin(4\eta) \tag{39}$$

$$a_{22} = \exp(-3\eta)[\cos(4\eta) - .75\sin(4\eta)], \tag{40}$$

where $\eta = (t_k - t_{k-1})$. The coefficient of u(k-1) is a vector $\underline{b}$ with the elements:

$$b_1 = \int_{t_{k-1}}^{t_k} 6.25\exp(-3\delta)\sin(4\delta)d\tau \tag{41}$$

$$b_2 = \int_{t_{k-1}}^{t_k} 6\exp(-3\delta)[\cos(4\delta) - .75\sin(4\delta)]d\tau, \tag{42}$$

where $\delta = (t_k - \tau)$. For a sampling interval of 0.01 seconds, $a_{11} = 0.99877$, $a_{12} = 0.040424$, $a_{21} = -0.058211$, $a_{22} = 0.94056$, $b_1 = 0.0012251$, and $b_2 = 0.058211$. Figure 3-7 contains a plot of the response of $x_1(k)$ and $x_2(k)$ for a unit step input.

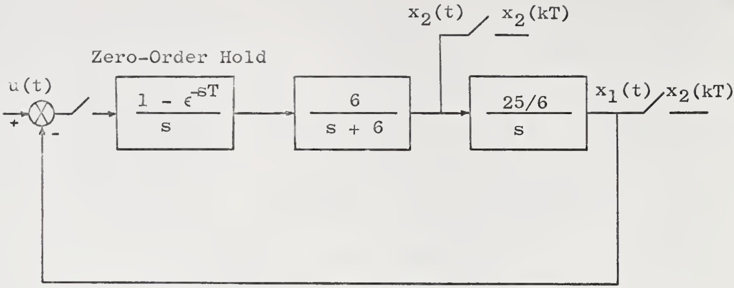

Figure 3-6 Second-Order System

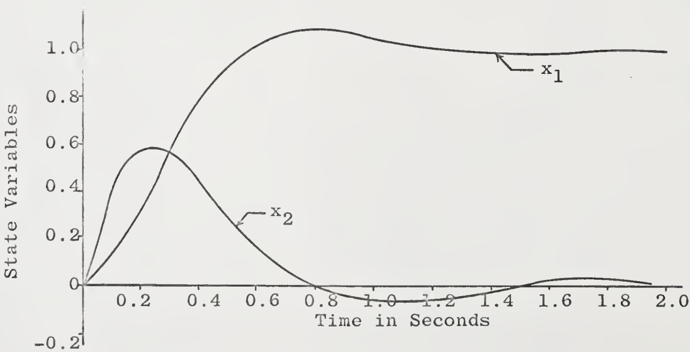

Figure 3-7 Unit Step Response

The perturbation matrix method was used to calculate the effects of changing the components of the matrix A. In this example, every element of A was changed the same amount by setting all elements of the matrix C equal to one. The resulting first-order change in $x_1(k)$ and $x_2(k)$ calculated by equation (14) for a step input are shown in figure 3-8. The change in $\underline{x}(k)$ for a given value of ϵ can be determined from figure 3-8 by multiplying the curves by ϵ.

The accuracy for a specific perturbation was determined by calculating the change in $\underline{x}(k)$ for various values of ϵc_{ij}, and then simulating the system with the elements of A actually changed by ϵc_{ij}. The actual changes in $x_1(k)$ and $x_2(k)$ from the simulations and the first-order changes calculated by equation (12) are very close. The exact changes given by equation (13) are identical to the changes observed in the simulations.

The sensitivity vectors were calculated for each element of A as a variable parameter. The elements of the sensitivity vectors for changes in a_{11} are shown in figure 3-9. To avoid confusion, the vectors for changes in the other elements of A are not included. The changes in $x_1(k)$ and $x_2(k)$, due to changing all elements of A by 0.02, were calculated by equation (17). The results are identical to the first-order changes predicted by the perturbation matrix approach.

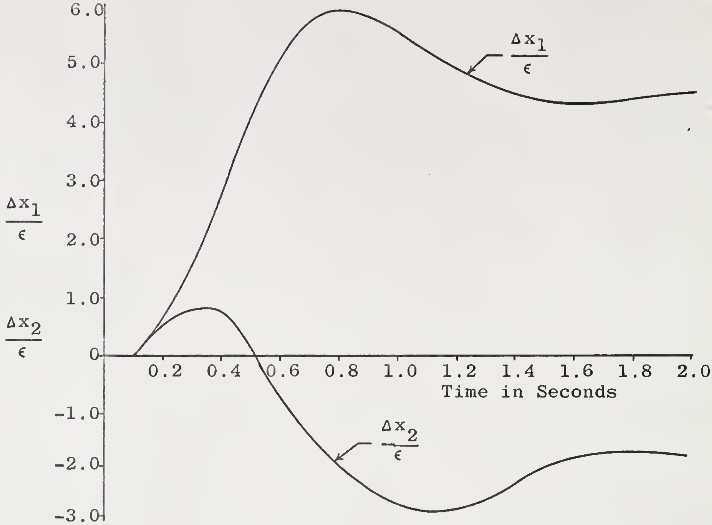

Figure 3-8 Generalized First-Order Change

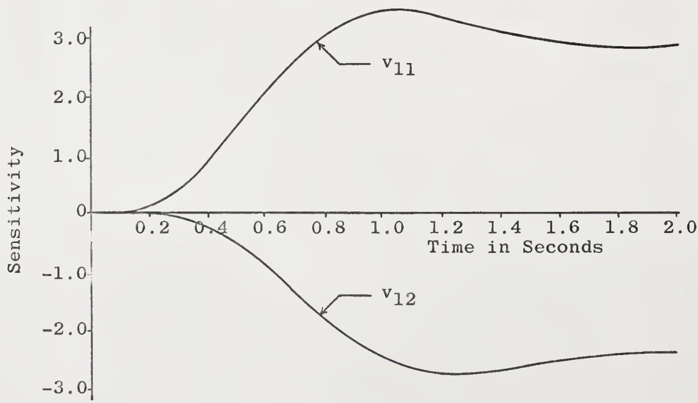

Figure 3-9 Sensitivity Vector Components

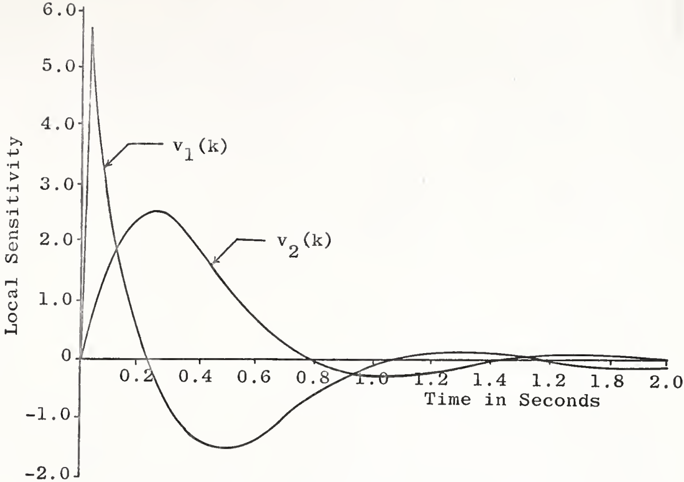

Figure 3-10 Local Sensitivity

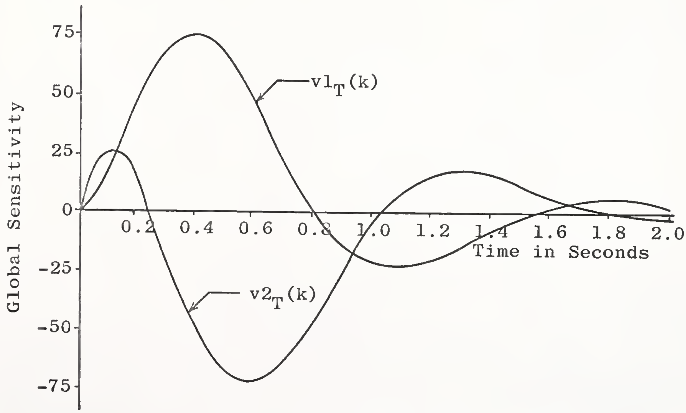

Figure 3-11 Global Sensitivity

The global and local sampling interval sensitivity vectors were calculated by equations (26) and (31) for a step input and T = 0.01 seconds. Figure 3-10 contains a plot of each element of the local sensitivity vector. The global sensitivity vector is shown in figure 3-11. Excellent results were obtained when the two vectors were checked by extrapolating values of $\underline{x}(k)$ about T = 0.01 for $\Delta T = 0.005$ and comparing them with values obtained from a simulation with T = 0.015.

Summary

Two approaches to parameter sensitivity for discrete systems have been presented. In each case, difference equations for the sensitivity function were developed and their calculation and use demonstrated.

The sensitivity to local and global changes in sampling interval were investigated and formulas for their calculation were given. Example problems were worked and the results compared with the changes observed when the sampling interval was actually changed.

Every method discussed was found to be accurate, and no problems were encountered in calculating and using the functions. Based on these observations, parameter and sampling interval sensitivity should be quite useful in the analysis and design of discrete systems.

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CHAPTER 4

VARIABLE INCREMENT OPTIMAL SAMPLING

Introduction

The purpose of this work is to develop a method for optimal adjustment of the sampling intervals of a sampled-data system. The object of adjustment is to obtain a low sampling rate, subject to a performance criterion that reflects system fidelity. This is appealing for such applications as time-sharing of a digital computer, control of discrete systems in which energy is to be conserved, and selection of a set of optimal sampling instants.

Adjustable sampling has been investigated by several approaches. Dorf, et al., [1] used the absolute value of the derivative of the error signal in Type I and Type II unity feedback systems to adjust the sampling rate. Gupta [2] used the ratio of the first and second derivative of the error signal to determine the sampling intervals. More recently, Tomović and Bekey [3] have used amplitude sensitivity to adjust sampling rate. The algorithm is based on the sensitivity of the system output to changes in output of a zero-order data hold for the error signal. They have also used the sensitivity of the system output to changes

in sampling interval to adjust the sampling rate [4]. The related problem of quantization error in hybrid systems has also been studied [5].

In this work, performance criteria for sampling interval adjustment will be selected, and error and state variable sampling interval sensitivity functions defined. Also included will be sampling interval formulas that incorporate sensitivity and the performance criteria to determine sampling intervals that maintain desired system fidelity.

Performance Criteria for Variable Increment Sampling

The performance criteria for sampling interval selection should reflect the effects of sampling interval variations on the discrete system state variables. The criterion should also include the type of data reconstruction, since this can affect the fidelity of the response.

Consider first the effects of sampling interval variations on the discrete system state variables at the sampling instants. If the discrete model of a continuous system is exact, then the values of the state variables are correct for any sampling interval size. However, if the discrete model is not exact, then the error between the continuous and discrete state variables at the sampling instants will, in general, depend upon the sampling intervals. The performance criteria should include this modeling error. This can be done analytically for linear

systems. For nonlinear systems, a combination of analytical and experimental methods must be used.

When the reconstructed response must meet certain specifications, then the error of the data reconstruction device must also be included. For a linear continuous system with an exact discrete model, the only error in the response is that of the reconstruction device. If an approximate discrete model is used, then both types of errors are present and must be considered. Since the data reconstruction device operates over a full sampling interval, a suitable criterion would be the magnitude or the mean- or integral-square value of the difference between the continuous and the reconstructed response over each sampling interval. If the exact solution is unknown, the response for very small sampling intervals could be used for the comparison. The exact solution will, of course, seldom be known in an on-line application.

Sampling Interval Sensitivity

In order to select the sampling interval, $T_{k-1} = t_k - t_{k-1}$, so that $\underline{x}(t_k)$ satisfies a sensitivity performance criterion, the effect of variations in T_{k-1} on $\underline{x}(t_k)$ must be known. This can be determined from local sampling interval sensitivity. If an approximate model is used, it would also be helpful to know the effects of variations in T_{k-1} on the error between continuous and discrete state

variables. This section will be devoted to an investigation of these effects.

Local state variable sampling interval sensitivity

The sensitivity of $\underline{x}(t_k)$ to variations in T_{k-1} is given by the local sampling interval sensitivity function. Variations in $T_{k-1} = t_k - t_{k-1}$ are obtained by perturbing t_k while t_{k-1} is held constant. A definition for local sampling interval sensitivity can be stated,

$$\underline{v}(k) = \frac{\partial \underline{x}(t_k)}{\partial t_k}, \tag{1}$$

where $\underline{v}(0) = 0$.

For the system represented by the equation

$$\underline{x}(t_k) = \underline{f}\{\underline{x}(t_{k-1}),\underline{u}(t_{k-1}),t_{k-1},t_k\}, \tag{2}$$

the sensitivity, $\underline{v}(k)$, can be obtained by differentiating with respect to t_k,

$$\underline{v}(k) = \frac{\partial \underline{f}\{\underline{x}(t_{k-1}),\underline{u}(t_{k-1}),t_{k-1},t_k\}}{\partial t_k}. \tag{3}$$

If the system is linear, sensitivity can be calculated from the equation

$$\underline{v}(k) = \frac{\partial A(t_{k-1},t_k)}{\partial t_k}\,\underline{x}(t_{k-1}) + \frac{\partial B(t_{k-1},t_k)}{\partial t_k}\,\underline{u}(t_{k-1}). \tag{4}$$

It should be noted that local sampling interval sensitivity gives the effect on the state vector $\underline{x}(t_k)$ of changing only T_{k-1}. It is not directly related to modeling

error at sampling instants nor reconstruction error between sampling instants. Unlike parameter sensitivity, it is not desirable to have sampling interval sensitivity approach or equal zero, for this would mean that the state vector never changes.

Local error sampling interval sensitivity

If the discrete model is approximate, it would be desirable to base sampling interval selection on some function of the error of the approximation. In order to determine sampling intervals that maintain certain error criteria, the sensitivity of the error to variations in sampling intervals is needed. This can be determined analytically for a linear system, using the exact discrete model and the discrete approximation used to obtain the approximate model.

As an example, consider the effect of using the rectangular rule to derive the discrete model of the linear continuous system,

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t), \tag{5}$$

where $\underline{x}(0) = \underline{x}_o$. If the rectangular rule for integration is used, an approximate discrete model for the system is,

$$\begin{aligned}\hat{\underline{x}}(t_k) = [I + T_{k-1}A(t_k,t_{k-1})]\hat{\underline{x}}(t_{k-1}) \\ + T_{k-1}B(t_k,t_{k-1})\underline{u}(t_{k-1}),\end{aligned} \tag{6}$$

where $T_{k-1} = t_k - t_{k-1}$ and I is the identity matrix.

An exact discrete model can be determined by discretizing the state transition equation,

$$\underline{x}(t) = \Phi(t,t_o)\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,\eta)B(\eta)\underline{u}(\eta)d\eta. \tag{7}$$

For the interval T_{k-1}, equation (7) becomes,

$$\underline{x}(t_k) = \Phi(t_k,t_{k-1})\underline{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \Phi(t_k,\eta)B(\eta)\underline{u}(\eta)d\eta. \tag{8}$$

Equation (8) can be further simplified if u(t) and B(t) are constant over the interval T_{k-1}. This implies that a zero-order sample and hold is being used at the input, and that B(t) is constant during sampling intervals. An additional simplification can be made by defining,

$$\Psi(t_k,t_{k-1}) = \int_{t_{k-1}}^{t_k} \Phi(t_k,\eta)d\eta. \tag{9}$$

Equation (8) can now be written,

$$\underline{x}(t_k) = \Phi(t_k,t_{k-1})\underline{x}(t_{k-1})$$
$$+ \Psi(t_k,t_{k-1})B(t_k,t_{k-1})u(t_{k-1}). \tag{10}$$

The error, $\underline{x}(t_k)$, in $\underline{x}(t_k)$, due to the approximate discrete model, is the difference,

$$\underline{\tilde{x}}(t_k) = \underline{x}(t_k) - \underline{\hat{x}}(t_k). \tag{11}$$

Using equations (6) and (10), an equation for the error is,

$$\underline{\tilde{x}}(t_k) = [\Phi(t_k,t_{k-1}) - I - T_{k-1}A(t_k,t_{k-1})]\underline{\hat{x}}(t_{k-1})$$

$$+ \Phi(t_k,t_{k-1})\underline{\tilde{x}}(t_{k-1})$$

$$+ [\Psi(t_k,t_{k-1}) - T_{k-1}I]B(t_k,t_{k-1})\underline{u}(t_{k-1}). \quad (12)$$

By varying t_k with t_{k-1} constant, the sampling interval T_{k-1} is changed, and the sensitivity of the error in $\underline{x}(t_k)$, due to variations in the sampling interval, can be defined as,

$$\underline{\tilde{v}}(t_k) = \frac{\partial\underline{\tilde{x}}(t_k)}{\partial t_k} . \quad (13)$$

At $k = 0$, the error, $\underline{\tilde{x}}(t_o)$, and the error sensitivity, $\underline{\tilde{v}}(0)$, are zero since the initial conditions are identical for both the exact and the approximate discrete models. Equation (12) can be used to determine the equation for the error sampling interval sensitivity,

$$\underline{\tilde{v}}(k) = \left[\frac{\partial\Phi(t_k,t_{k-1})}{\partial t_k} - A(t_k,t_{k-1})\right]\underline{\hat{x}}(t_{k-1})$$

$$- T_{k-1}\frac{\partial A(t_k,t_{k-1})}{\partial t_k}\underline{\hat{x}}(t_{k-1}) + \frac{\partial\Phi(t_k,t_{k-1})}{\partial t_k}\underline{\tilde{x}}(t_{k-1})$$

$$+ \Psi(t_k,t_{k-1})\frac{\partial B(t_k,t_{k-1})}{\partial t_k}\underline{u}(t_{k-1})$$

$$+ \left[\frac{\partial\Psi(t_k,t_{k-1})}{\partial t_k} - I\right]B(t_k,t_{k-1})\underline{u}(t_{k-1}). \quad (14)$$

If the allowable error, $\underline{\tilde{x}}(t_k)$, is specified, equation (12) could be used to determine the sampling intervals

required to keep $\underline{x}(t_k)$ within tolerance. However, in this work, sampling interval equations will be based on the error sampling interval sensitivity of equation (14). It should be noted that the error formulas were derived for an approximate discrete model based on the rectangular integration formula. Similar formulas can be derived for other approximations.

Since it is not always possible to determine an exact discrete model, the analysis of the effects of an approximate discrete model must often be determined by experimental means.

Variable Increment Sampling Based on Sensitivity

The accuracy of the response of certain components of the state vector might be more important than others, and sampling interval calculation should reflect these requirements. One way to do this is to weigh the components of the sensitivity vector. Each component of the local sampling interval sensitivity vector, $\underline{v}(k)$, is the sensitivity of the corresponding component in the state vector to changes in sampling interval. Another way of incorporating different performance requirements would be to specify a performance criterion for each element of the state vector. For each interval, a sampling increment could be calculated from each element of the sensitivity vector and the corresponding performance criterion. The desired interval would be selected from the resulting set.

Sampling interval calculation and selection could possibly be simplified somewhat by using the quadratic form, $\underline{v}^T Q \underline{v}$, for sampling interval calculations. There are several advantages. Only one sampling interval would be calculated for each sampling period, and there would be no need to select an interval from a set of n sampling intervals. The different performance requirements on the elements of the state vector can be maintained by selecting the elements of the matrix Q to give different weighting to the elements of the sensitivity vector.

Linear system simulation with an exact discrete model

Consider the simulation of a linear continuous system with an exact discrete model derived from the state transition equation. The error due to discretization of the input will be neglected. Under these conditions, the values of the discrete system state variables at the sampling instants are identical to the response of the continuous system. The only error in the response of the discrete model is between sampling instants, and it depends upon the sampling interval and the type of data reconstruction used. In this section, formulas that use sampling interval sensitivity to calculate sampling intervals will be derived. The sampling interval formulas will incorporate performance criteria that constrain the difference between the continuous system response and reconstructed response of the discrete model.

To derive sampling interval formulas, consider first zero-order hold reconstruction. The zero-order hold maintains the value of the state at the preceding sampling instant until a new value is obtained at the next sampling instant. For an exact discrete model, the continuous and sampled response for a zero-order hold is shown in figure 4-1.

The derivation of sampling interval formulas will be based on two approximations. First, it will be assumed that the continuous response can be approximated by a straight-line over a single interval. This is shown in figure 4-1. For this assumption, the error of zero-order hold data reconstruction is approximated by the shaded area of figure 4-1. The other assumption is that $\underline{x}(t_k)$ can be determined from the first two terms of a Taylor's series,

$$\underline{x}(t_k) \cong \underline{x}(t_{k-1}) + [\underline{v}(k)](t_k - t_{k-1}), \tag{15}$$

where $\underline{v}(k)$ is the local sampling interval sensitivity.

In order to maintain the error within specified limits, the performance criterion should be related to the error triangle of figure 4-1. The maximum difference, the integral of the difference, or the average of the difference squared, could be used. Since all three are based on the shaded triangle of figure 4-1, they are identical. If the performance criterion is the maximum difference, then the sampling interval T_{k-1} should be selected so that,

$$\left|x_i(t_k) - x_i(t_{k-1})\right| \leq m_i \qquad (i = 1,\cdots,n). \tag{16}$$

The elements, m_i, of the performance vector $\underline{m}$ will be selected to maintain the desired accuracy of the corresponding state variable. For the approximations used in this derivation, the integral of the difference is related to the maximum difference by $id_i = 1/2T_{k-1}m_i$. The maximum difference is related to the average of the difference squared by $ads_i = 1/3T_{k-1}(m_i)^2$.

A sampling interval equation that maintains the inequality of equation (16) can be derived with the use of equation (15). The i^{th} sampling interval equation for T_{k-1} is,

$$T_{(k-1)_i} \cong \frac{m_i}{|v_i(k)|}, \tag{17}$$

where T_{k-1} must be selected from the set of n intervals. If the minimum value is used,

$$T_{k-1} = \min T_{(k-1)_i} \qquad (i = 1,\cdots,n), \tag{18}$$

then all elements of the state vector should be within the limits of the performance criteria. Different performance requirements on certain elements of the state vector can be handled by using different values for m_i or by weighting the elements of the sensitivity vector $v_i(k)$.

If the quadratic form $\underline{v}^T(k)Q\underline{v}(k)$ is used, a different approach must be used to derive a sampling interval equation. To investigate the use of the quadratic form to

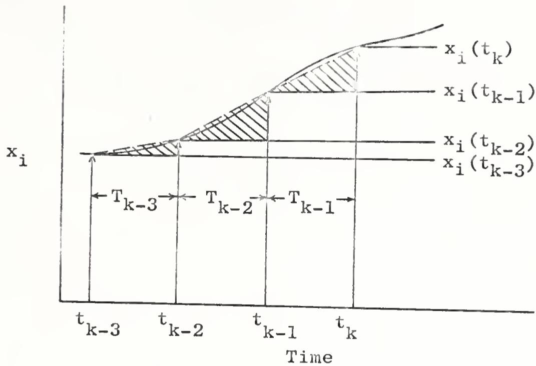

Figure 4-1 Zero-Order Data Reconstruction

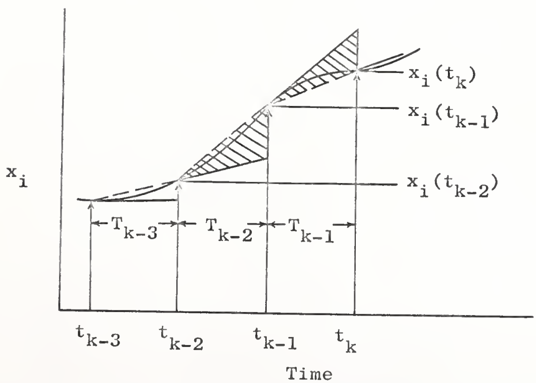

Figure 4-2 First-Order Data Reconstruction

determine sampling intervals, consider a second-order system and let Q be a diagonal matrix with diagonal elements q_{11} and q_{22}. For these conditions, the quadratic form is

$$\underline{v}^T(k)Q\underline{v}(k) = q_{11}v_1^2(k) + q_{22}v_2^2(k). \tag{19}$$

If the approximation of equation (15) is used, a sampling interval equation can be derived

$$T_{k-1} \cong \left[\frac{q_{11}[x_1(t_k) - x_1(t_{k-1})]^2}{\underline{v}^T(k)Q\underline{v}(k)} + \frac{q_{22}[x_2(t_k) - x_2(t_{k-1})]^2}{\underline{v}^T(k)Q\underline{v}(k)}\right]^{1/2} . \tag{20}$$

Once the desired difference between $x_i(t_k)$ and $x_i(t_{k-1})$ is established, equation (20) can be used to calculate the sampling interval. The use of the quadratic form for sampling interval calculations will be illustrated with an example problem.

If a first-order hold is used for data reconstruction, the response during a given interval is the straight-line extension of the sampled values from the two preceding sampling instants. This is shown in figure 4-2. The reconstructed sampled response during the interval $(t_k - t_{k-1})$ is given by the equation

$$x_i(t_{k-1} + \eta) = \left[\frac{x_i(t_{k-1}) - x_i(t_{k-2})}{T_{k-2}}\right](\eta) + x_i(t_{k-1}), \tag{21}$$

where $\eta \leq (t_k - t_{k-1})$.

In deriving sampling interval equations for first-order hold reconstruction, the same approximations used in the zero-order hold case will be assumed. That is, the continuous response will be approximated by a straight-line over a single interval. Also, it is assumed that $\underline{x}(t_k)$ is given by the approximation of equation (15). The difference between the assumed continuous response and approximate response is represented by the shaded triangle of figure 4-2. Based on these assumptions, the maximum difference will occur when $\eta = (t_k^- - t_{k-1})$. Thus, maximum error occurs when the reconstructed response of equation (21) is

$$x_i(t_k^-) = \left[\frac{x_i(t_{k-1}) - x_i(t_{k-2})}{T_{k-2}}\right] T_{k-1} + x_i(t_{k-1}). \qquad (22)$$

The value of the maximum difference for the interval $(t_k - t_{k-1})$ is $x_i(t_k) - x_i(t_k^-)$.

If the performance criterion is based on limiting the maximum difference, a sampling interval formula can be derived for the criterion,

$$\left| x_i(t_k) - x_i(t_k^-) \right| \leq m_i \qquad (i = 1, \cdots, n). \qquad (23)$$

Using equations (15),(22) and (23), the sampling interval is

$$T_{(k-1)_i} = \frac{m_i}{\left| v_i(k) - \dfrac{x_i(t_{k-1}) - x_i(t_{k-2})}{T_{k-2}} \right|}, \qquad (24)$$

where $i = 1,\cdots,n$. From the approximation for the $\underline{x}(t_k)$ in equation (15), the sensitivity is

$$v_i(k-1) \cong \frac{x_i(t_{k-1}) - x_i(t_{k-2})}{T_{k-2}} , \tag{25}$$

and equation (20) can be changed to

$$T_{(k-1)_i} \cong \frac{m_i}{\left|v_i(k) - v_i(k-1)\right|} \qquad (i = 1,\cdots,n). \tag{26}$$

The actual interval T_{k-1} is selected from the set on n intervals calculated from equation (26).

The quadratic form $\underline{v}^T(k)Q\underline{v}(k)$ can also be used to determine the sampling interval for an exact discrete model with first-order data reconstruction. Consider a second-order system and use equations (15),(19) and (22). The resulting sampling interval equation is

$$T_{k-1} = \left[\frac{q_{11}\{x_1(t_k)-x_1(t_{k-1})]^2 - [x_1(t_k^-)-x_1(t_{k-1})]^2\}}{\underline{v}(k)^T Q\underline{v}(k) - \underline{v}^T(k-1)Q\underline{v}(k-1)} + \frac{q_{22}\{[x_2(t_k)-x_2(t_{k-1})]^2 - [x_2(t_k^-)-x_2(t_{k-1})]^2\}}{\underline{v}(k)^T Q\underline{v}(k) - \underline{v}^T(k-1)Q\underline{v}(k-1)}\right]^{1/2} . \tag{27}$$

Note that in order to use equation (27), both the maximum difference between the sampled and the continuous response, and the maximum difference of the state variables, over an interval, must be specified. For this reason, it may not be as useful as other formulations.

A more general approach for determining the reconstructed response of a discrete system would be to use

a fractional-order hold [6]. For a fractional-order hold, the reconstructed response during the interval $(t_k - t_{k-1})$, is given by the equation

$$x_i(t_{k-1} + \eta) = (F)\,\frac{x_i(t_{k-1}) - x_i(t_{k-2})}{T_{k-2}}\,(\eta) + x_i(t_{k-1}), \tag{28}$$

where $\eta = (t_k - t_{k-1})$ and $0 \leqq F \leqq 1$. If $F = 0$, equation (28) is for a zero-order hold, and for $F = 1.0$, equation (28) is identical to the first-order hold of equation (21). If the fractional-order hold is used, the sampling interval equation is

$$T_{(k-1)_i} \cong \frac{m_i}{\left| v_i(k) - Fv_i(k-1) \right|}\,, \tag{29}$$

where $i = 1,\cdots,n$, and $0 \leqq F \leqq 1$. If the quadratic form is used, equation (27) will have the terms $q_{11}[x_1(t_k^-) - x_1(t_{k-1})]^2$ and $q_{22}[x_2(t_k^-) - x_2(t_{k-1})]^2$ divided by F^2.

<u>Linear system simulation with an approximate discrete model</u>

In a previous section, it was pointed out that both modeling and reconstruction errors are present if an approximate model is being used. In this section, sampling interval formulas that constrain these errors will be derived.

First, consider only the error in the discrete system variables at the sampling instants. In figure 4-3, the response for the discrete model is shown along with the

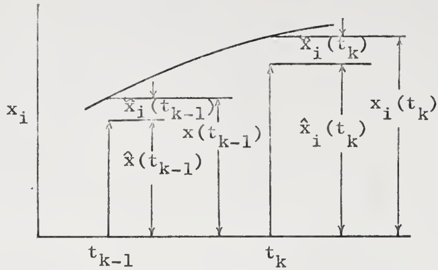

Figure 4-3 Approximate Model Response

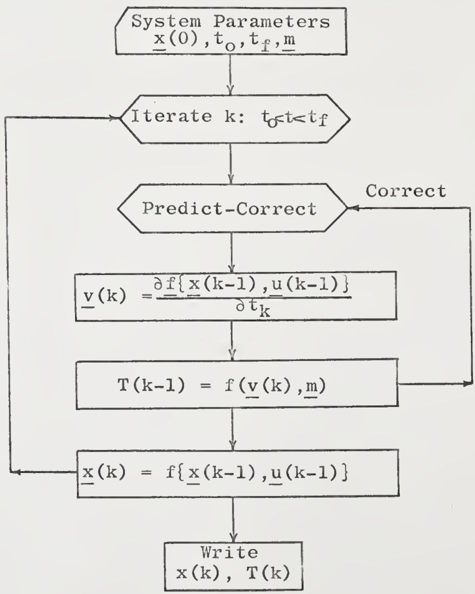

Figure 4-4 Calculation Procedure

exact or approximate continuous response. A suitable criterion would be to constrain the magnitude of the error by the relationship,

$$\left|x_i(t_k)\right| \leq m_i . \tag{30}$$

For a linear system, an equation for the error sensitivity, similar to equation (13), could be derived for the approximate discrete system. Then the approximation,

$$\tilde{x}_i(t_k) \cong \tilde{x}_i(t_{k-1}) + T_{k-1}\tilde{v}_i(k), \tag{31}$$

could be used to derive a relationship for the sampling intervals,

$$\left|\tilde{x}_i(t_{k-1}) + T_{k-1}\tilde{v}_i(k)\right| \leq m_i \qquad (i = 1,\cdots,n). \tag{32}$$

The error sensitivity could also be used to derive sampling interval formulas that constrain the change in error,

$$\left|\tilde{x}_i(t_k) - \tilde{x}_i(t_{k-1})\right| \leq m_i . \tag{33}$$

The resulting sampling interval formula is

$$T_{(k-1)_i} \cong \frac{m_i}{\left|\tilde{v}_i(k)\right|} \qquad (i = 1,\cdots,n). \tag{34}$$

If the difference between the reconstructed response and the exact or approximate continuous response is to be constrained, the sampling interval formulas must include both data reconstruction and modeling errors. With zero-order hold reconstruction, figure 4-3 indicates that a

suitable criterion for sample interval selection would be the constraint

$$\left|[\hat{x}_i(t_k) - \hat{x}_i(t_{k-1})] + \tilde{x}_i(t_k)\right| \leq m_i . \tag{35}$$

Substitution of the approximations of equations (15) and (31) into equation (35) gives

$$\left|[\hat{v}_i(k) + \tilde{v}_i(k)]T_{k-1} + \tilde{x}_i(t_{k-1})\right| \leq m_i . \tag{36}$$

If first-order data reconstruction is used, the performance criterion takes the form of the constraint

$$\left|[\hat{x}_i(t_k) - \hat{x}_i(t_k^-)] + \tilde{x}_i(t_k)\right| \leq m_i . \tag{37}$$

Equations (15),(22) and (31) can be used to determine an approximate sampling interval relationship,

$$\left|[\hat{v}_i(k) - \hat{v}_i(k-1) + \tilde{v}_i(k)]T_{k-1} + \tilde{x}_i(t_{k-1})\right| \leq m_i . \tag{38}$$

If a fractional-order hold is used, the relationship is

$$\left|[\hat{v}_i(k) - F\hat{v}_i(k-1) + \tilde{v}_i(k)]T_{k-1} + \tilde{x}_i(t_{k-1})\right| \leq m_i , \tag{39}$$

where $0 \leq F \leq 1.0$.

Nonlinear system simulation

If the continuous system is nonlinear, the discrete form of the system will, in general, be an approximate model. Therefore, both modeling and reconstruction errors will be present. Both types of simulation errors have

already been discussed for linear system simulation. State variable sampling interval sensitivity was used to derive sampling interval formulas that constrained the reconstruction error. The modeling error of linear system simulation, with an approximate model, was limited by sampling intervals derived from an error sampling interval sensitivity.

Since state variable sampling interval sensitivity is also applicable for nonlinear systems, it can be used for adjusting sampling intervals to constrain the reconstruction error. However, error sensitivity is not readily available for nonlinear systems, and, therefore, it cannot be used to determine sampling intervals that constrain modeling error. Instead, state variable sampling interval sensitivity will be used to calculate sampling intervals. The resulting modeling error will be controlled by adjusting parameters of the sampling interval calculator. This is not as appealing as the variable increment sampling technique worked out for linear system simulation. However, the sampling efficiency can be improved without seriously impairing the accuracy of the simulation.

For nonlinear system simulation, with an approximate discrete model, equation (29) can be used to calculate sampling intervals. The values of $\underline{m}$ and F are then adjusted experimentally to achieve the desired system fidelity. The process will be illustrated with example problems.

Calculation Procedure for System Simulation

The order of the calculation sequence is very important. Several problems were encountered in applying the formulas that have been derived. In order to provide additional insight into sample adjustment, these problems and their solutions will be presented along with the calculation procedure of simulation.

The first problem encountered was with calculating the interval between $\underline{x}(k-1)$ and $\underline{x}(k)$. Before any of the formulas for sampling interval can be used to calculate T_{k-1}, the sensitivities, $\underline{v}(k)$, and $\underline{\tilde{v}}(k)$, must be known. However, $\underline{v}(k)$, the sensitivity of $\underline{x}(k)$ to changes in T_{k-1}, and $\underline{\tilde{v}}(k)$, the sensitivity of $\underline{\tilde{x}}(k)$ to changes in T_{k-1}, are functions of t_k, which depends on T_{k-1}. The easiest way to avoid this problem is to use values from the preceding interval to calculate $\underline{v}'(k)$ and $\underline{\tilde{v}}'(k)$. For example, $\frac{\partial A(t_{k-2},t_{k-1})}{\partial t_{k-1}}$ and $\frac{\partial B(t_{k-2},t_{k-1})}{\partial t_{k-1}}$ are used in place of $\frac{\partial A(t_{k-1},t_k)}{\partial t_k}$ and $\frac{\partial B(t_{k-1},t_k)}{\partial t_k}$ in equation (4). This procedure was used in actual simulation and gave good results for some example problems.

An improvement can be obtained by using the predicted values of T_{k-1} to calculate values for $\frac{\partial A(t_{k-1},t_k')}{\partial t_k}$ and $\frac{\partial B(t_{k-1},t_k')}{\partial t_k}$, and then use these values to recalculate $\underline{v}(k)$, $\underline{\tilde{v}}(k)$, and T_{k-1}. This predictor-corrector scheme

could be repeated to further improve the accuracy of T_{k-1}. An additional refinement can be obtained by comparing the prediction with the correction. A large difference would indicate a need to reduce the sampling interval.

The order of calculation for simulation is as follows:

- Calculate $\underline{v}(k)$ and $\underline{\tilde{v}}(k)$ using values from the previous interval.
- Use sensitivity to calculate T_{k-1} as outlined in the preceding paragraphs.
- Calculate $\underline{x}(k)$ with equation (2).
- Repeat the cycle for the next interval.

The method is not self-starting, and the first interval must be preset to start the simulation.

Another problem encountered in developing the variable sampling technique was brought about by sudden changes in the input. When the system is operating in a state of low sensitivity, the sampling intervals will be large. If a sudden change in input occurred just after a large sampling interval had been selected, a considerable portion of the ensuing transient response might be missed. This can be avoided by monitoring the input and sampling immediately after any change in input greater than a suitable threshold. Figure 4-4 contains a flow chart for the calculation procedure.

System Analysis for Variable Increment Sampling

If the techniques developed in the preceding sections are used to determine sampling rate, the analysis of the resulting system is quite involved. A block diagram for the complete system is shown in figure 4-5. Block A represents the actual system. The sensitivities are determined in block B. Block B would also represent the comparison of the prediction and correction if this is included. Block C calculates the sampling interval from the sensitivities (predicted and corrected), and block D limits the size of the sampling intervals. The limits on the size of the sampling interval will be discussed further in the next paragraph. Block E provides T_o to start simulation, begins correction if the predictor-corrector scheme is used, and initiates sampling if a sudden change in input occurs.

In discussing limits on sampling interval size, consider first the lower limit. Since the purpose of varying the sampling increment is to increase sampling efficiency, the sampling rate is only as high as is necessary to obtain an acceptably performing system. The minimum value of sampling interval is that sampling interval beyond which further reduction in size is not considered worthwhile. Certainly, the highest sampling frequency (smallest sampling interval) must be several times the highest frequency in the input and output of the system [6]. In actual practice, the minimum is

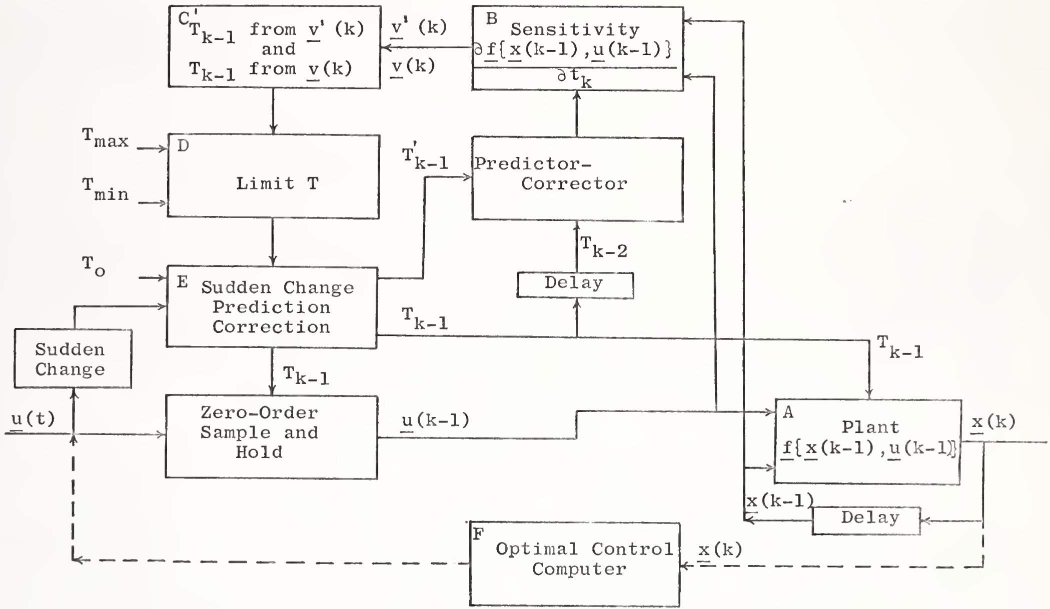

Figure 4-5 System Diagr n

determined by the system bandwidth and the required accuracy. Therefore, the lower limit on sampling interval cannot be set in general.

The maximum allowable sampling interval for a closed-loop system is usually determined by steady-state stability requirements. However, if an approximate model is being used, the accuracy of the approximation will require that the maximum allowable sampling interval be somewhat below the stability limit. Thus, the upper limit on sampling interval can be set only after careful study of the particular system.

The introduction of variable increment sampling completely changes the stability considerations for the sampled-data system. In addition to the stability of the system of block A, the sampling interval selection loop stability (blocks B, C, D, and E) must also be studied. In addition, there is certainly coupling between loops.

The system stability (block A) can be studied by opening the sampling interval selection loop and determining limits on the allowable values of sampling interval for stability. For a linear system, the allowable values of sampling interval can be determined by root-locus techniques. However, a procedure based on the "second-method" of Lyapunov is more general, since it applies to nonlinear systems. The use of the "second-method" to determine asymptotic stability of difference equations

has been studied [7,8,9], and it has also been applied to frequency and pulse-width modulated sampling systems [2, 10,11].

The stability theorem can be stated [10] as follows: a sufficient condition for asymptotic stability (in the large) of the vector difference equation, $\underline{x}(k) = \underline{f}\{\underline{x}(k-1)\}$, is the existence of $V(\underline{x})$, a scalar function of the state variables such that:

1. $V(0) = 0$
2. $V(\underline{x}) \geqq 0$ when $x \neq 0$
3. $V[\underline{x}(k)] < V[\underline{x}(k-1)]$ for $k < K$, K finite
4. $V(\underline{x})$ is continuous in $\underline{x}$
5. $V(\underline{x}) \rightarrow \infty$ when $\underline{x} \rightarrow \infty$,

where $V(\underline{x})$ is a Lyapunov function for the system.

The stability of the sampling interval selection loop is more difficult to determine. Probably the best approach is computer simulation.

Variable Increment Sampling for Optimal Control

The application of variable increment optimal sampling to an optimal controller is very appealing. Sampling interval sensitivity could be used to determine the sampling interval for both the system and the optimal control computer. A system of this type is shown in figure 4-5. The optimal control computer (block F) is shown with dotted connections.

The scheme in figure 4-5 could possibly be improved by determining the sampling interval sensitivity of both the optimal controller and the system. The two sensitivities could then be used to determine a sampling interval for the entire system. The resulting sampling interval should be the "best" for both the system and the optimal controller.

There are several ways to determine a sampling interval from the two sensitivity vectors. One method is to use the quadratic form $\underline{v}_a^T(k)Q\underline{v}_a(k)$, where $\underline{v}_a(k)$ is a n + m column vector constructed from the sensitivity vectors of the system and the controller. Another method is to weigh the elements of the sensitivity vectors. Both methods have already been discussed. The calculation procedure is as follows:

- Determine the sampling interval sensitivity vectors. There are n components for the system and m for the optimal controller.
- From the n + m sensitivity components, calculate a sampling interval.
- Use the resulting sampling increment for the next interval of both system and controller.
- Repeat the procedure.

Although this procedure requires m additional sensitivity calculations, the possibility of obtaining the "best" sampling increment for both system and controller may justify the extra calculations.

Example Problems

The techniques discussed in the preceding sections will be illustrated in the following example problems.

Problem 1

For the system shown in figure 4-6, the form of the difference equation is,

$$x(k) = bx(k-1) + cu(k-1). \tag{40}$$

If the state transition equation is discretized, the exact (except for the effect of discretizing the input) model will have the coefficients

$$b = \exp\{-a(t_k - t_{k-1})\}, \tag{41}$$

and

$$c = 1/a[1 - \exp\{-a(t_k - t_{k-1})\}]. \tag{42}$$

The input selected for checking the variable increment technique was

$$u(t) = 1.0, \quad 0 \leqq t \leqq 5.0 \text{ and } 6.0 \leqq t \leqq 7.0, \tag{43}$$

and

$$u(t) = 0, \tag{44}$$

for all other values of t. This is shown with a dashed line in figure 4-7. This particular input was selected

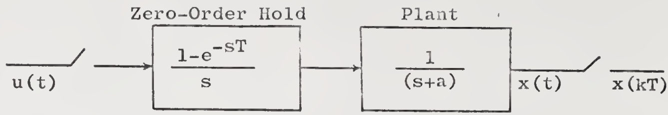

Figure 4-6 First-Order System

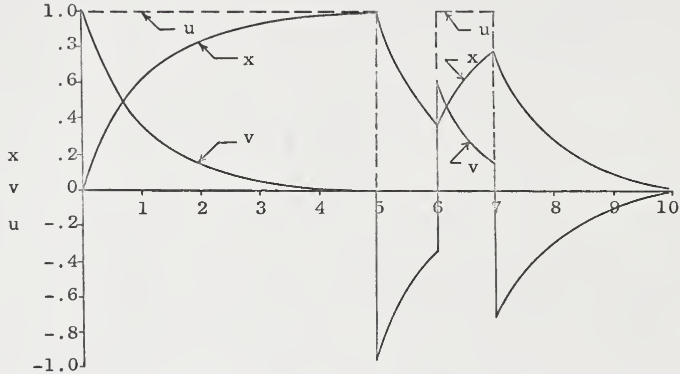

Figure 4-7 Response and Sensitivity

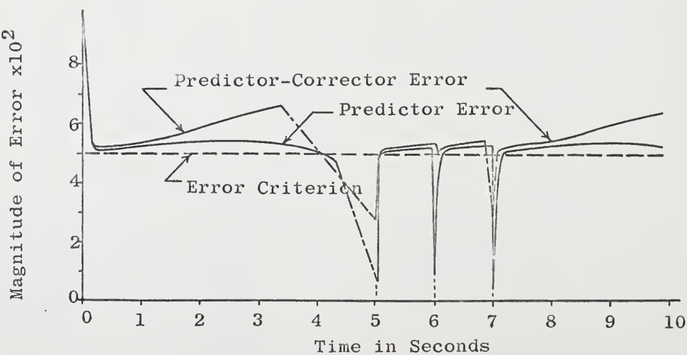

Figure 4-8 Data Reconstruction Error: Zero-Order Hold

since it can eliminate the error of discretizing the input and, at the same time, provide a rather severe test of the sampling adjustment process. The response, x, and the local sampling interval sensitivity, v, for an initial condition $x(0) = 0$, are shown in figure 4-7.

For this exact discrete model, the only error is that of data reconstruction. The magnitude of the reconstruction error of a zero-order hold is shown in figure 4-8. Sampling intervals were determined from equation (17) with $m = 0.05$. The figure includes curves for both predictor and predictor-corrector calculating methods.

The reconstruction error for the predictor scheme is closer to the desired value, 0.05, than that of the predictor-corrector method. The reason for this can be seen in figure 4-7. Sensitivity is a decreasing function, and if data from the preceding interval are used, a smaller sampling interval is calculated. If the sensitivity is corrected, the sampling interval will be larger, and as a result, the error is increased. If the sensitivity were an increasing function, the predictor-corrector scheme would give smaller intervals and the reconstruction error would be reduced.

The large initial error shown in figure 4-8 is due to the size of the first sampling interval. The first interval is preset, and, therefore, the initial error can be controlled. In figure 4-8, the initial sampling interval was selected to give a large initial error to show that the

error is reduced at the next sampling interval. The sudden decreases in the error at 5, 6 and 7 seconds are due to the small sampling intervals selected by the sudden change provision. Also, note that the sign of the reconstruction error changes at 5, 6 and 7 seconds.

Figure 4-9 contains a plot of sampling interval versus time for several different values of reconstruction error magnitude. The discontinuities at 5, 6 and 7 seconds are due to sudden changes in the input.

Figure 4-10 presents the trade-off between the number of samples required to cover 10 seconds of the response and the magnitude of the reconstruction error. The curve shown is for predictor sampling. The number of samples required is also strongly dependent on the nature of the input.

Since the main objective of variable increment sampling is to save computer time, it is necessary to compare variable and fixed increment sampling for the same magnitude of reconstruction error. For a zero-order hold and variable increment sampling, 130 samples were required to cover 10 seconds of the response. The maximum magnitude of the reconstruction error was 0.02635. In order to maintain the same maximum error with fixed interval sampling, 476 samples would be required to cover 10 seconds. The ratio of the number of samples is 0.273.

The reduction in the number of sampling intervals does not give a complete evaluation. Each iteration of variable increment sampling requires more computer time

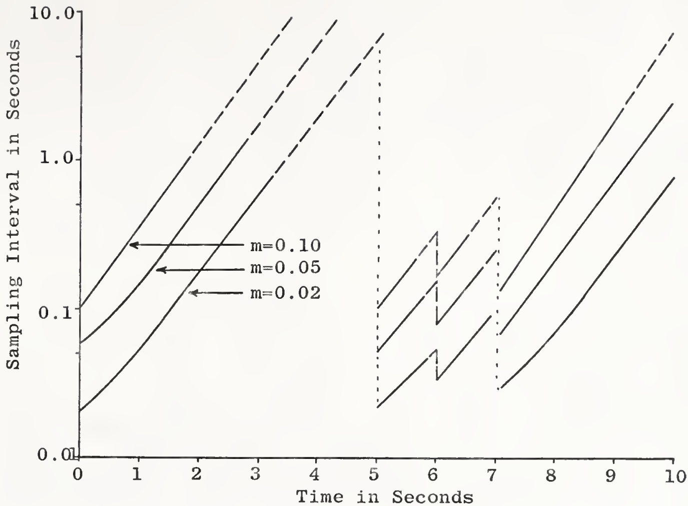

Figure 4-9 Sampling Interval: Zero-Order Hold

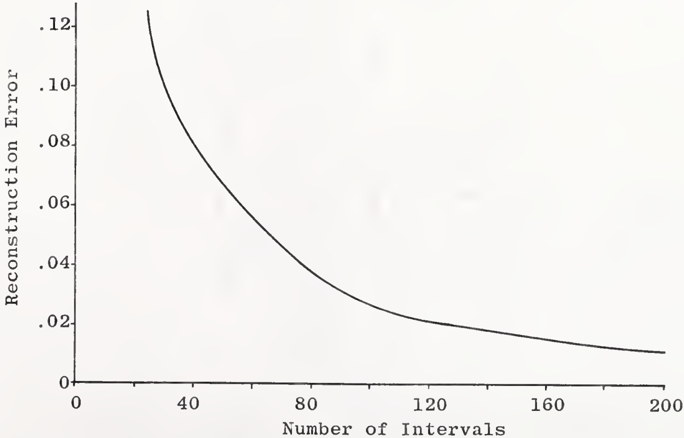

Figure 4-10 Zero-Order Reconstruction Error vs. Number of Intervals

than an iteration for fixed interval sampling. For this particular system, the ratio of calculation time is approximately 3.11. Thus, a more accurate figure of merit, for the first 10 seconds, would be the product of the two ratios, 0.273 x 3.11 = 0.848. If the same input is used, and the first 20.0 seconds considered, variable increment sampling is even more appealing. The combined figure of merit drops to 0.438.

The effect of varying F in fractional-order hold reconstruction is shown in figure 4-11. For values of F greater than 0.5, the sampling intervals were alternately large and small and the results are not useful.

The system was simulated with an approximate model based on the rectangular rule for integration. The difference equation for the system is

$$\hat{x}(k) = [1-a(t_k-t_{k-1})]\hat{x}(k-1) + (t_k-t_{k-1})u(k-1). \tag{45}$$

The sampling interval sensitivity equation is

$$\hat{v}(k) = -a\hat{x}(k-1) + u(k). \tag{46}$$

The difference equation for the error is

$$\begin{aligned}\tilde{x}(k) = {} & [\exp\{-a(\eta)\} - 1 + a(\eta)]\tilde{x}(k-1) \\ & + \exp\{-a(\eta)\}\hat{x}(k-1) \\ & + [1/a(1-\exp\{-a(\eta)\}) - \eta]u(k-1),\end{aligned} \tag{47}$$

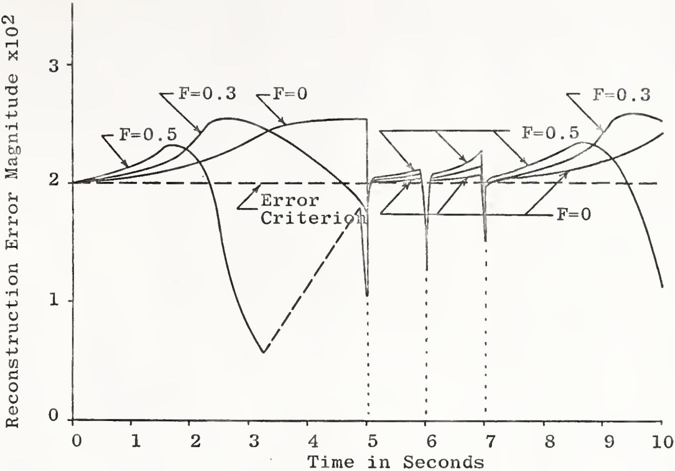

Figure 4-11 Reconstruction Error: Fractional-Order Hold

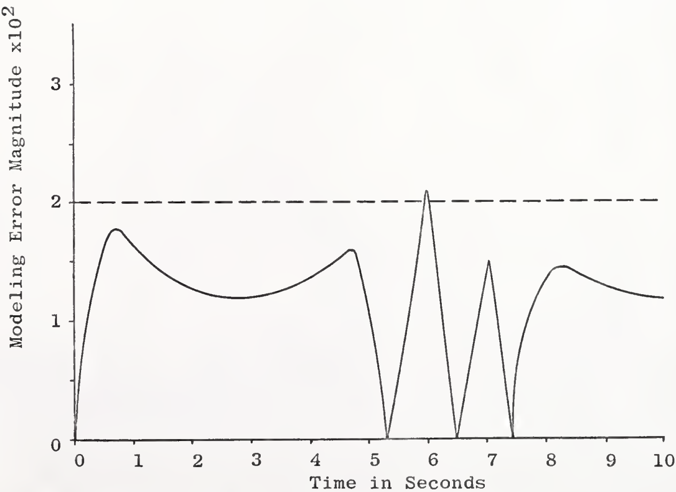

Figure 4-12 Modeling Error of Approximate Model

and the error sampling interval sensitivity is

$$\hat{\tilde{v}}(k) = a[1-\exp\{-a(\eta)\}]\tilde{x}(k-1)$$

$$-a[\exp\{-a(\eta)\}]\hat{x}(k-1)$$

$$+ [\exp\{-a(\eta)\} - 1]u(k-1), \tag{48}$$

where $\eta = t_k - t_{k-1}$.

The error sensitivity of equation (48) was used in equation (34) to calculate sampling intervals for the approximate model. The magnitude of the resulting modeling error is shown in figure 4-12 for m = 0.02. Figure 4-14 contains a plot of sampling interval size versus time. Sampling interval size is varied by 20 to 1 to keep the magnitude of the modeling error within the desired level, 0.02.

In figure 4-13, the magnitude of the algebraic sum of the reconstruction and modeling errors is shown for two values of F, the fraction of the data-hold circuit. Sampling intervals were determined from equation (39). The discontinuities at t = 5, 6 and 7 seconds are due to sudden changes in the input.

The corresponding curves for sampling interval size are shown in figure 4-14. It is interesting to compare the F = 0 sampling interval curve of figure 4-14 with the m = 0.02 curve of figure 4-9. If the modeling and reconstruction errors are of opposite sign, the sampling interval curve of figure 4-14 gives a larger sampling

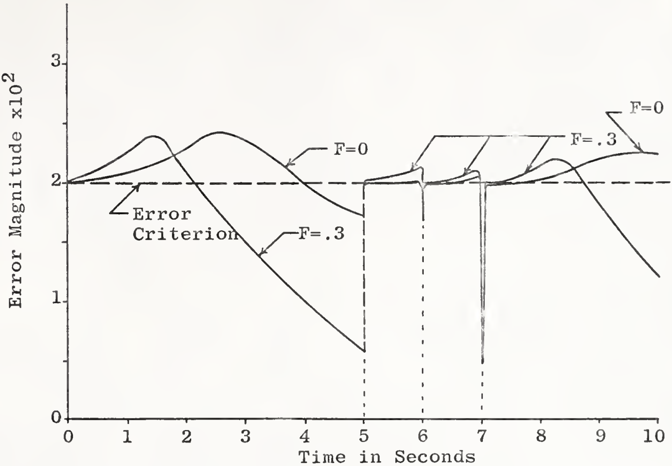

Figure 4-13 Reconstruction and Modeling Error

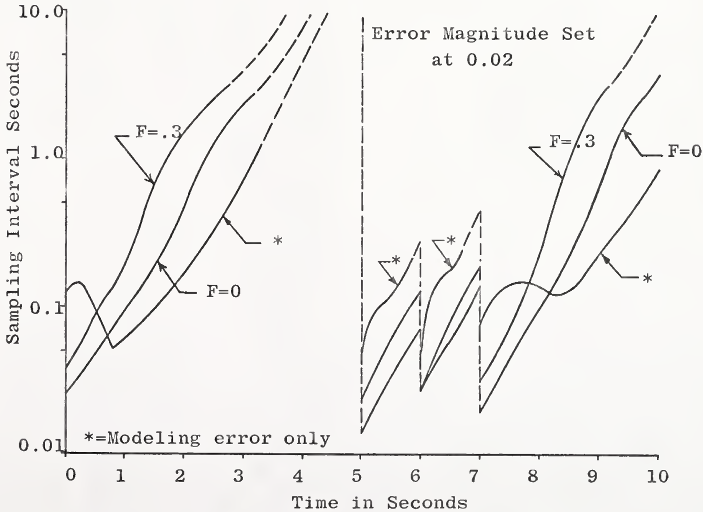

Figure 4-14 Sampling Interval: Approximate Model

interval than the curve of figure 4-9. When both errors are of the same sign, the curve of 4-14 gives lower values of sampling interval.

Problem 2

Variable increment sampling has also been applied to the system of figure 4-15. The exact difference equation for the system is

$$\underline{x}(k) = A\underline{x}(k-1) + \underline{b}u(k-1), \tag{49}$$

where the components of A are,

$$a_{11} = \exp(-3T)[\cos(4T) + .75\sin(4T)] \tag{50}$$

$$a_{12} = (25/24)\exp(-3T)\sin(4T) \tag{51}$$

$$a_{21} = -1.5\exp(-3T)\sin(4T) \tag{52}$$

$$a_{22} = \exp(-3T)[\cos(4T) - .75\sin(4T)]. \tag{53}$$

The elements of the vector $\underline{b}$ are

$$b_1 = 0.25[4.0 - \exp(-3T)\{3\sin(4T) + 4\cos(4T)\}] \tag{54}$$

and

$$b_2 = 1.5\exp(-3T)\sin(4T). \tag{55}$$

The local sampling interval equation is

$$\underline{v}(k) = C\underline{x}(k-1) + \underline{d}u(k-1), \tag{56}$$

where the components of C are,

$$c_{11} = -\exp(-3T)[6.25\sin(4T)] \tag{57}$$

$$c_{12} = \exp(-3T)[(25/6)\cos(4T) - (25/8)\sin(4T)] \tag{58}$$

$$c_{21} = \exp(-3T)[4.5\sin(4T) - 6\cos(4T)] \tag{59}$$

$$c_{22} = -\exp(-3T)[6\cos(4T) + 1.75\sin(4T)]. \tag{60}$$

The elements of $\underline{d}$ are

$$d_1 = 6.25\exp(-3T)\sin(4T) \tag{61}$$

and

$$d_2 = \exp(-3T)[6\cos(4T) - 4.5\sin(4T)]. \tag{62}$$

Variable increment sampling was investigated for the input

$$u(t) = 1.0 \quad 0 \leqq t \leqq 1.0 \text{ and } 1.4 \leqq t \leqq 1.6, \tag{63}$$

and

$$u(t) = 0, \tag{64}$$

for all other values of t. The input is shown with dashed lines in figure 4-16. This input is similar to the input used in problem 1. Figure 4-16 also contains the response curves, x_1 and x_2, for the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. The sensitivities, v_1 and v_2, are shown in figure 4-17.

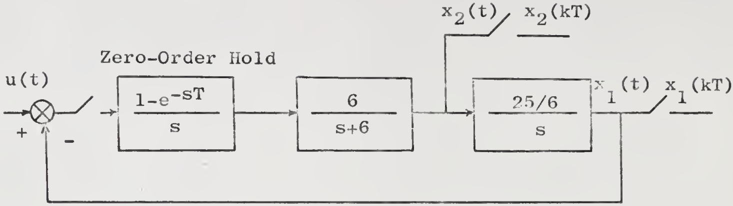

Figure 4-15 Second-Order System

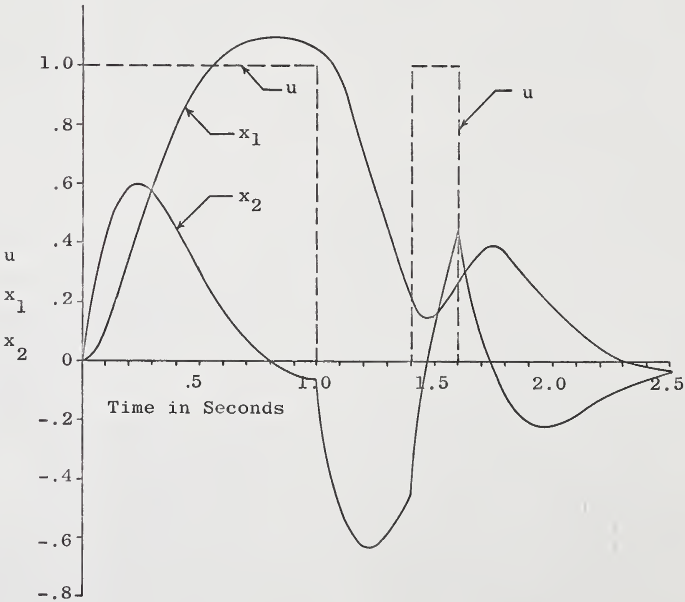

Figure 4-16 Input and Response

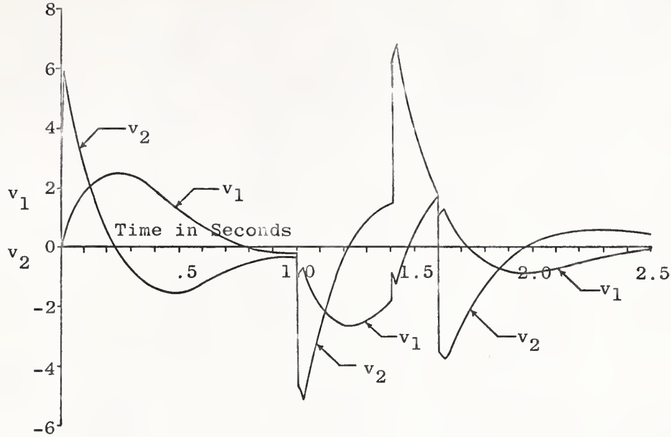

Figure 4-17 Sensitivity

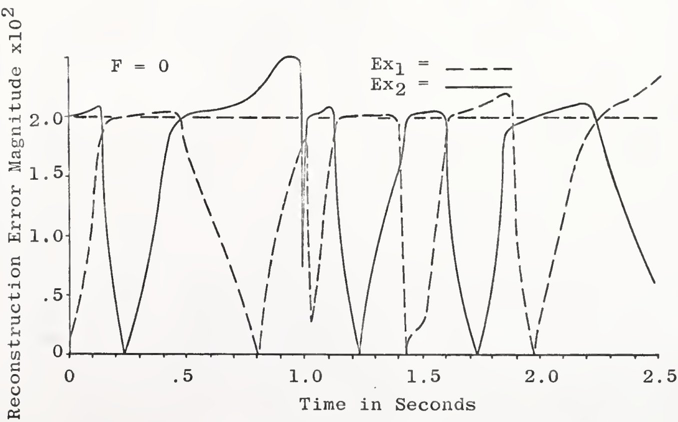

Figure 4-18 Reconstruction Error

For this problem, equation (29) was used to calculate a sampling interval from each component of the sensitivity vector. The smaller interval was used in simulation. The resulting reconstruction error is shown in figure 4-18. Note that the reconstruction error, for one or the other state variables, is always near the desired level, $m_1 = m_2 = 0.02$.

The corresponding sampling interval curve is shown in figure 4-20. The discontinuities at 1.0, 1.4 and 1.6 seconds are caused by sudden changes in the input. Also, the slope of the sampling interval curve changes when the error limiting magnitude switches from one state variable to the other state variable.

The use of the quadratic form for sampling interval calculation is illustrated in figure 4-19 for $q_{11} = q_{22} = 1.0$ and $m_1 = m_2 = 0.02$. The dashed curve of figure 4-20 gives sampling interval size versus time for quadratic sampling. Several different values of q and $\underline{m}$ were used to determine their effect on the response. If q_{11} is larger than q_{22}, the error is reduced for x_1 and increased for x_2. Larger values of q increase the size of the sampling intervals. Increasing $\underline{m}$ also gives larger sampling intervals and an increase in reconstruction error.

For a maximum error magnitude of 0.0262, variable increment sampling reduced the number of samples needed to cover 5 seconds from 1430 to 233. This represents a sampling reduction ratio of 0.163. This, combined with

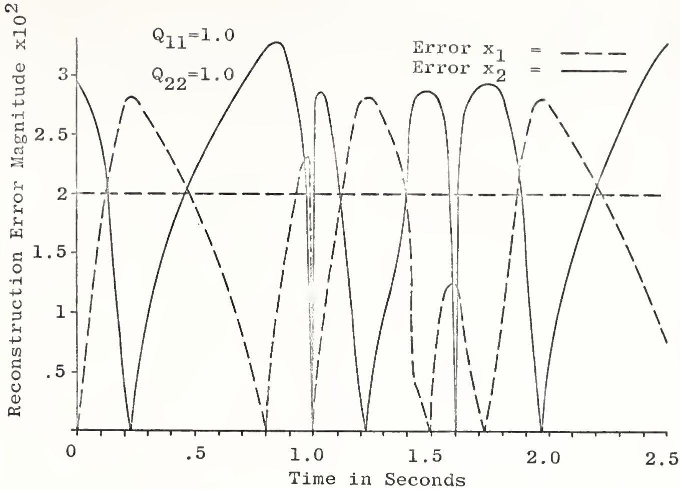

Figure 4-19 Reconstruction Error

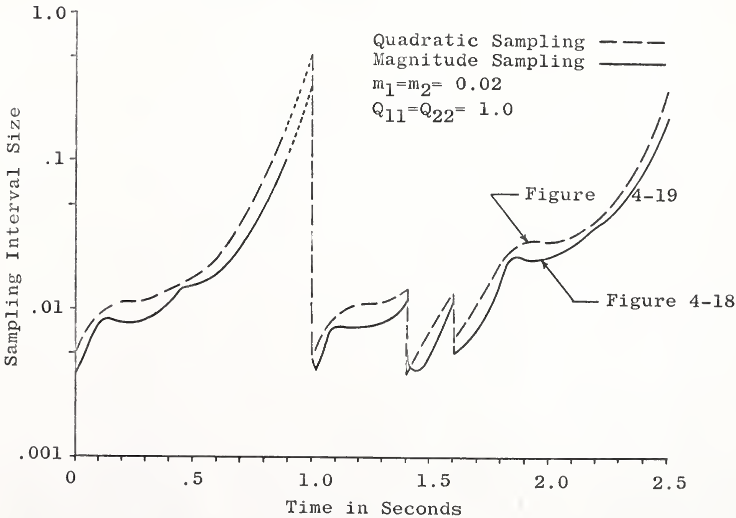

Figure 4-20 Sampling Interval

the iteration computing time ratio of approximately 3.22, gives an over-all utilization factor of 0.525.

The system was also simulated with an approximate model based on the rectangular rule for integration. The difference equation for the approximate system is

$$\underline{\hat{x}}(k) = E\underline{\hat{x}}(k-1) + \underline{r}u(k-1), \tag{65}$$

where the matrix E is

$$E = \begin{bmatrix} 1 & 25/6(t_k-t_{k-1}) \\ -6(t_k-t_{k-1}) & [1 - 6(t_k-t_{k-1})] \end{bmatrix}, \tag{66}$$

and the vector $\underline{r}$ is

$$\underline{r} = \begin{bmatrix} 0 \\ 6(t_k-t_{k-1}) \end{bmatrix}. \tag{67}$$

The difference equation for the error is

$$\underline{\tilde{x}}(k) = S\underline{\hat{x}}(k-1) + A\underline{\tilde{x}}(k-1) + \underline{h}u(k-1), \tag{68}$$

where the matrix S is

$$S = \begin{bmatrix} (a_{11} - 1) & [a_{12} - (25/6)(t_k-t_{k-1})] \\ [a_{21} + 6(t_k-t_{k-1})] & [a_{22} - 1 + 6(t_k-t_{k-1})] \end{bmatrix}. \tag{69}$$

The matrix A has already been defined. The vector $\underline{h}$ is

$$\underline{h} = \begin{bmatrix} b_1 \\ [b_2 - 6(t_k-t_{k-1})] \end{bmatrix}, \tag{70}$$

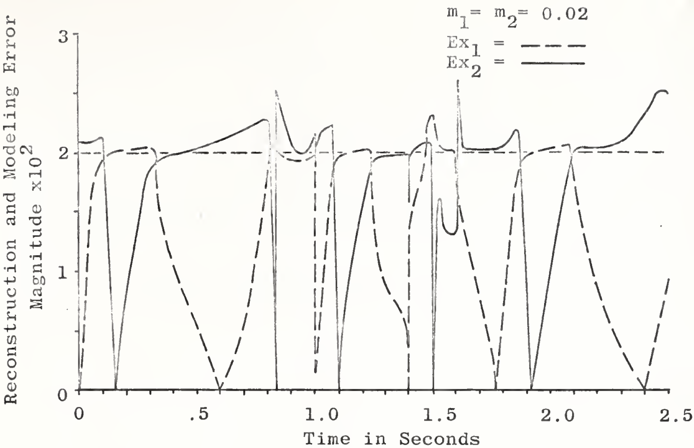

Figure 4-21 Reconstruction and Modeling Error

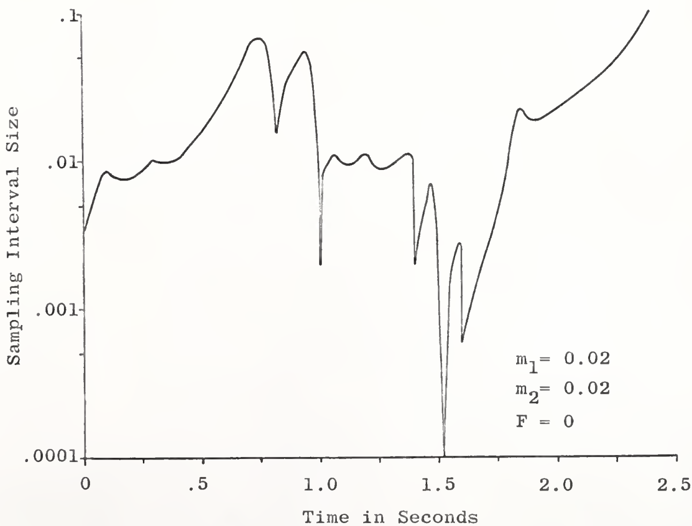

Figure 4-22 Sampling Intervals

where b_1 and b_2 were defined in equations (54) and (55).

Equation (39) was used to calculate sampling intervals for the approximate model. The algebraic sum of the reconstruction and modeling error is shown in figure 4-21. The corresponding sampling interval curve is shown in figure 4-22. The sampling interval ranged from 0.0001 to 0.1719 seconds in order to maintain the error curves shown in figure 4-21. This is the greatest range observed in the course of this work. Various values of $\underline{m}$ and F were used, and it was found that values of F, greater than 0.5, produced large and small sampling intervals alternately. The values for $\underline{m}$ depend upon the model. Smaller values of $\underline{m}$ are possible with more accurate models.

Problem 3

An interesting application of variable increment sampling, for a nonlinear system, is the optimal start up of a thermal nuclear reactor. If the precursors are represented by a single average group, the kinetic equations for the reactor are [12]

$$\dot{n} = [(\rho - \beta)/\Lambda]n + \lambda c, \tag{71}$$

and

$$\dot{c} = (\beta/\Lambda)n - \lambda c, \tag{72}$$

where,

n = neutron flux density ($1/cm^3$)

ρ = reactivity (input)

c = average precursor density ($1/cm^3$)

β = fraction of precursors formed

Λ = neutron lifetime (seconds)

λ = precursor decay constant (1/second).

The values of the system parameters are assumed to be: $\beta = 0.0064$, $\Lambda = 0.001$ and $\lambda = 0.10$.

Figure 4-23 contains the optimal operating curves that take the reactor from 0.5 KW to 5.0 KW in one second and minimize the cost function

$$J = 1/2\int_{o}^{1.0} \rho^2 dt. \tag{73}$$

The initial conditions are $n = 0.5$ and $c = 32.0$.

For digital simulation, the approximate difference equations

$$\hat{n}(k) = [(t_k - t_{k-1})\{(\rho - \beta)/\Lambda\} + 1]\hat{n}(k-1)$$
$$+ \lambda(t_k - t_{k-1})\hat{c}(k-1), \tag{74}$$

and

$$\hat{c}(k) = (t_k - t_{k-1})(\beta/\Lambda)\hat{n}(k-1)$$
$$+ [1 - (t_k - t_{k-1})\lambda]\hat{c}(k-1), \tag{75}$$

were obtained with the rectangular integration rule. For this approximate model, the sensitivity equations are

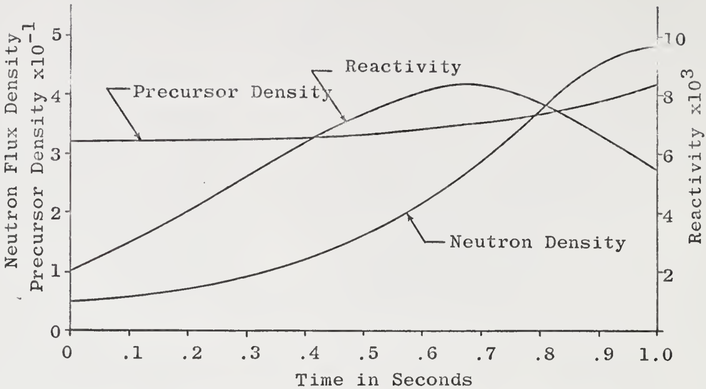

Figure 4-23 Reactor Optimal Response

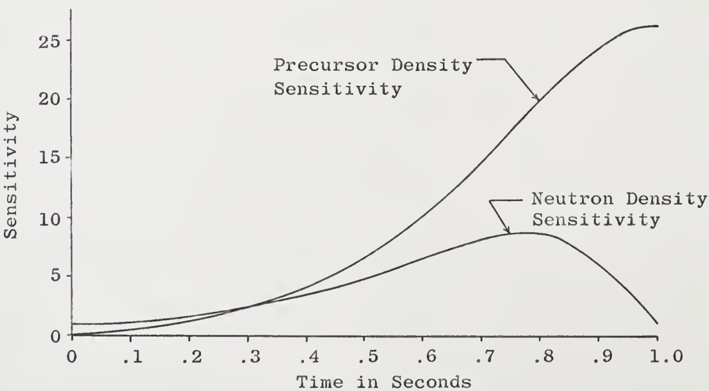

Figure 4-24 Sensitivity

$$\hat{v}_n(k) = [(\rho - \beta)/\Lambda]\hat{n}(k-1) + \lambda\hat{c}(k-1), \tag{76}$$

and

$$\hat{v}_c(k) = (\beta/\Lambda)\hat{n}(k-1) - \lambda\hat{c}(k-1). \tag{77}$$

The sensitivity functions are shown in figure 4-24. Note that, although the precursor concentration is much greater than the neutron density, the two sensitivity functions are almost equal for a large range of the response.

The system was first simulated with small fixed intervals of 0.001 seconds. This solution was used for comparison. Modeling error was defined as the difference between the reference solution and the response for variable increment sampling. For each sampling period, equation (17) was used to determine a sampling interval from both sensitivity functions. The smaller interval was used in variable increment sampling.

The magnitude of the modeling error, for this simulation, is shown with solid curves in figure 4-25. The magnitude of the precursor concentration error is misleading since the largest value is only 0.662 per cent, as compared to 2.68 per cent error for neutron density.

During the interval $0 \leq t \leq 1.0$, the ratio of the largest to the smallest sampling interval is 26.2 to 1. This is shown with the solid curve of figure 4-26. Variable increment sampling required 190 samples to cover the interval. Comparison with the reference solution of 1000

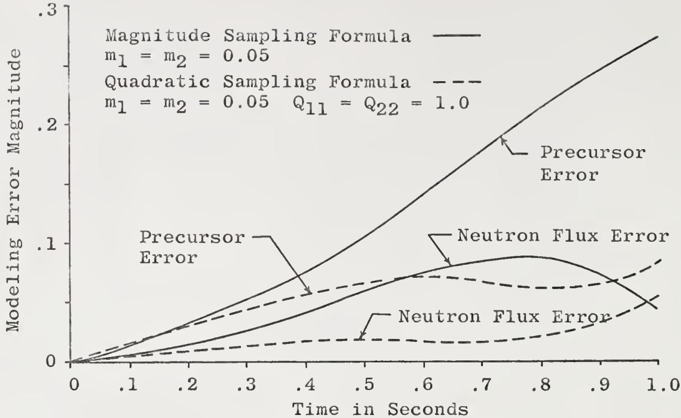

Figure 4-25 Modeling Error

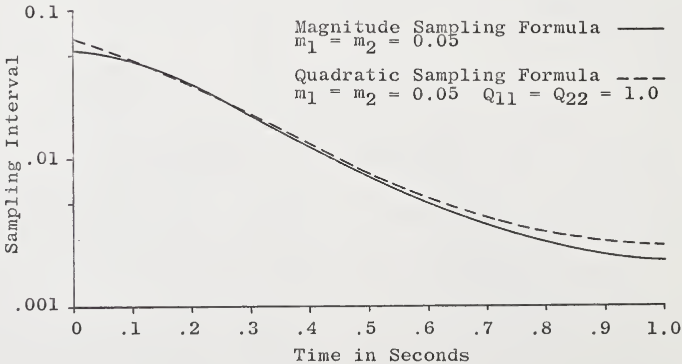

Figure 4-26 Sampling Interval

fixed sampling intervals indicates that an 81 per cent reduction in the number of sampling intervals produced a maximum error of 2.86 per cent in neutron density and 0.66 per cent in precursor concentration.

An over-all figure of merit can be obtained by including the ratio of iteration computing time. For this system the ratio is 2.65 for each iteration. Multiplication by the sampling reduction of 0.19 gives a utilization factor of 0.504.

Sampling intervals were also calculated from the quadratic form of equation (20). The dashed curves of figure 4-25 and 4-26 give modeling error and sampling interval size for $m_1 = m_2 = 0.05$ and $q_{11} = q_{22} = 1.0$. The sampling interval size was varied by 23.2 to 1. The maximum error is 0.21 per cent for precursor concentration and 1.09 percent for neutron density. This is better than the results obtained when sampling intervals were determined from equation (17).

The effects of varying q and $\underline{m}$ were investigated and results similar to those of problem 2 were obtained.

Problem 4

As an illustration of variable increment sampling for a time varying system, consider the equation

$$\dot{x} = x(a \sin(t) - 1), \tag{78}$$

where $x(0) = 1.0$. The exact solution for this equation is

$$x(t) = \exp\{a - a\cos(t) - t\}. \tag{79}$$

Using the rectangular rule for integration, an approximate discrete model is

$$\hat{x}(k) = \hat{x}(k-1)[1 + (t_k - t_{k-1})(a\sin(t_k) - 1)], \tag{80}$$

and the sensitivity equation is

$$\hat{v}(k) = \hat{x}(k-1)[a\sin(t_k) - 1]. \tag{81}$$

The response, $\hat{x}(k)$, and the sensitivity, $\hat{v}(k)$, are shown in figure 4-27 for $a = 1.5$.

The sampling interval sensitivity of equation (81) was used in equation (17) to determine sampling intervals for variable increment sampling. The magnitude of the resulting modeling error, for $m = 0.02$ and $a = 1.5$, is shown by the solid curve of figure 4-28. The corresponding sampling curve is shown in figure 4-29.

The ratio of the largest to the smallest sampling interval is 19.35. In order to evaluate the utility of the variable increment method, consider a fixed sampling interval solution with $T = 0.0232$, which is the smallest interval used in the variable sampling simulation. Using this sampling interval, 216 samples would be required to cover 5 seconds. For variable increment sampling, only 96 samples are needed, and the sample reduction ratio is 0.445.

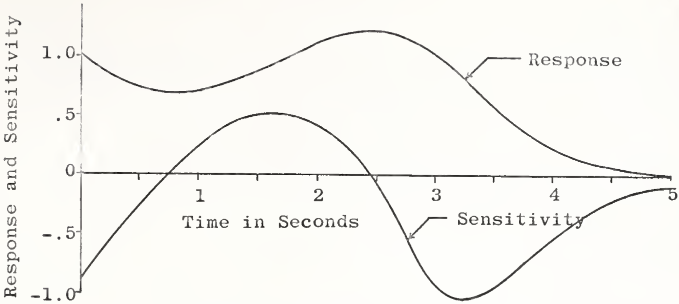

Figure 4-27 Response and Sensitivity

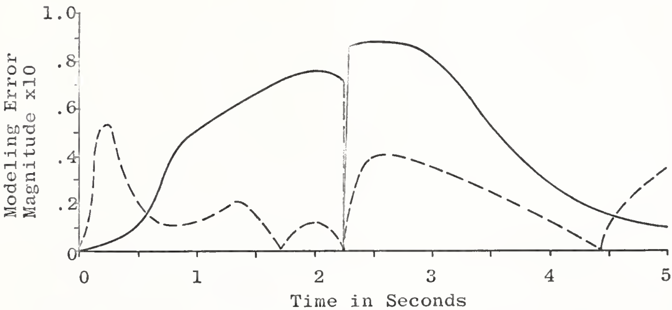

Figure 4-28 Modeling Error

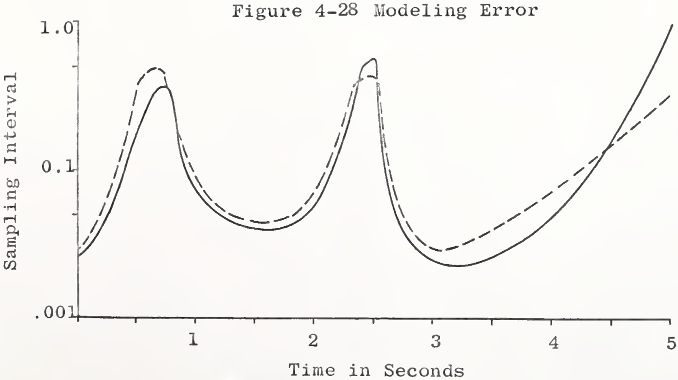

Figure 4-29 Sampling Interval

An over-all utilization factor can be determined by multiplying 0.445 by the iteration computing time ratio which is 2.23 for this system. The result, 0.99, is not favorable.

In an attempt to improve the utilization factor, an improved model was developed. The difference equation for the model is

$$x(k) = x(k-1)\left[\frac{1+\eta}{1-\eta}\right], \tag{82}$$

where

$$\eta = (1/2)(t_k - t_{k-1})(a \sin(t_k) - 1). \tag{83}$$

The magnitude of the error is shown by the dashed curve of figure 4-28. The corresponding sampling interval curve is shown in figure 4-29.

When the new model is used for variable increment sampling, the error is reduced and fewer sampling intervals are required. The utilization factor for the improved model is 0.875.

Problem 5

To illustrate the use of variable increment sampling for a nonlinear system, consider Van der Pol's equation,

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0. \tag{84}$$

Using phase variables and an approximation based on the

rectangular rule for integration, the approximate difference equations

$$\hat{x}_1(k) = (t_k - t_{k-1})\hat{x}_2(k-1) + \hat{x}_1(k-1), \tag{85}$$

and

$$\hat{x}_2(k) = \epsilon(t_k - t_{k-1})[1 - \hat{x}_1(k-1)^2]\hat{x}_2(k-1)$$
$$- (t_k - t_{k-1})\hat{x}_2(k-1) + \hat{x}_2(k-1), \tag{86}$$

were developed to represent equation (84). For this model, the sensitivity equations are

$$\hat{v}_1(k) = \hat{x}_2(k-1), \tag{87}$$

and

$$\hat{v}_2(k) = \epsilon[1 - \hat{x}_1(k-1)^2]\hat{x}_2(k-1) - \hat{x}_1(k-1). \tag{88}$$

For comparison, the approximate model was simulated with small fixed sampling intervals of 0.001 seconds. The initial conditions were $x_1(0) = 0$ and $x_2(0) = 0.5$, and the value for ϵ was 6.0. The response curves, shown in figure 4-31, for $x_1(k)$ and $x_2(k)$ were used as "reference" solutions for variable increment sampling. The corresponding sensitivity curves are shown in figure 4-32.

Variable increment sampling was employed for several different values of $\underline{m}$ and q. The magnitude of the error, for quadratic sampling with $q_{11} = q_{22} = 1.0$ and $m_1 = m_2 = .05$, is shown in figure 4-33. The large error indicates that

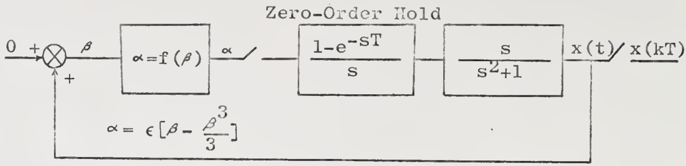

Figure 4-30 Model for Van der Pol's Equation

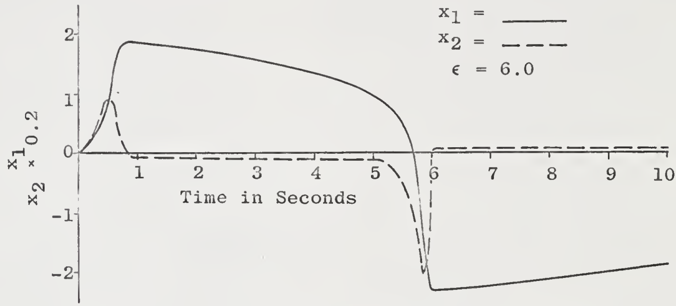

Figure 4-31 Response of Van der Pol's Equation

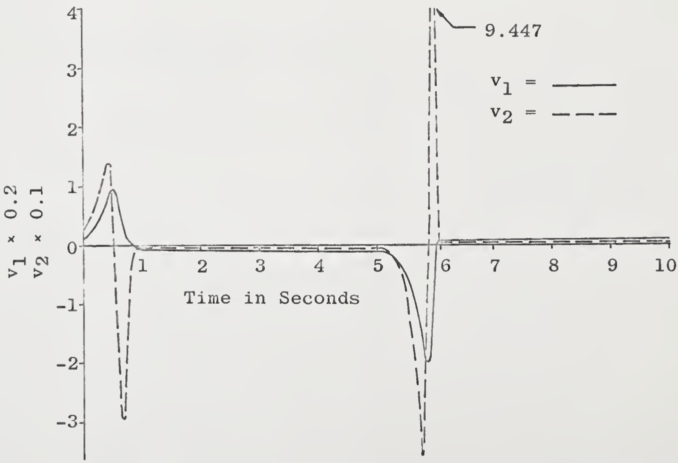

Figure 4-32 Sensitivity for Van der Pol's Equation

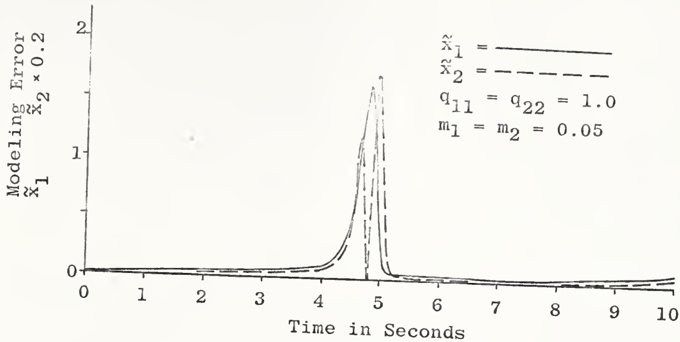

Figure 4-33 Modeling Error

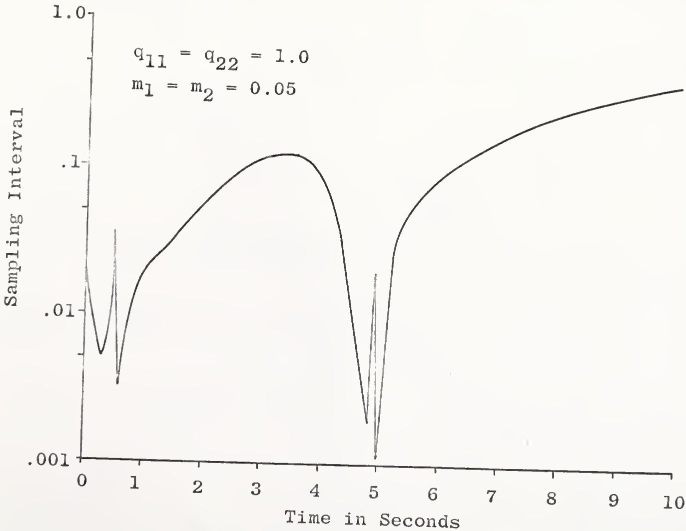

Figure 4-34 Sampling Interval

this is a poor model for large sampling intervals. The corresponding sampling interval curve is shown in figure 4-34.

The ratio of the largest to the smallest sampling interval was 361. Variable increment sampling required 703 samples to cover the interval $0 \leq t \leq 10$ seconds. The reference solution required 10,000.

In order to reduce the error of the variable sampling response, a better model for Van der Pol's equation was employed. The improved model is shown in figure 4-30. The difference equations for the simulation are

$$\alpha(k) = \epsilon[\hat{x}(k-1) - (1/3)\hat{x}(k-1)^3], \tag{89}$$

$$\begin{aligned} \hat{x}_2(k) = {} & \alpha(k) - [1 + \cos(\eta)]\alpha(k-1) \\ & + [\alpha(k-2) + 2\hat{x}_2(k-1)]\cos(\eta) \\ & - \hat{x}_2(k-2), \end{aligned} \tag{90}$$

and

$$\begin{aligned} \hat{x}_1(k) = {} & (1/2)\eta[\hat{x}_2(k) + \hat{x}_2(k-1)] \\ & + \hat{x}_1(k-1), \end{aligned} \tag{91}$$

where

$$\eta = t_k - t_{k-1}.$$

Using the improved model, the magnitude of the error was reduced by 40 per cent. The sampling interval curve

was not appreciably changed. Using the reference solution for comparison, the sampling reduction ratio, multiplied by the iteration computing time ratio, gives an over-all utilization factor of 0.22. This represents a considerable saving in computation time. However, there is a modest error at the switching points.

Summary

A method for adjusting the sampling intervals of a sampled-data system has been presented. The technique, based on sensitivity, determines sampling intervals in a manner such that the fidelity of the discrete system response is maintained at a specified level.

For linear systems, both modeling and reconstruction errors are considered. An error sampling interval sensitivity is developed and used with local sampling interval sensitivity to derive sampling interval formulas. The formulas constrain the magnitude of the algebraic sum of the two errors or the magnitude of either error individually.

For nonlinear systems, only modeling error is considered, and a combined analytical and experimental approach is necessary. Sampling interval formulas are based on sensitivity. However, it is necessary to adjust the parameters of the equation to give the desired response.

System stability was discussed, and it was pointed

out that an upper limit must be placed on the sampling interval to avoid instability. The results of simulation indicate that, for reasonable accuracy requirements, sampling intervals calculated by sensitivity are considerably below the stability limit. In most cases, the limiting value would be much lower than that required for stability, and selection would be partially dictated by accuracy requirements.

The techniques discussed were illustrated with several example problems. In each case, sampling efficiency and computer utilization improved with variable increment sampling. It should be emphasized that the actual improvement will, in all cases, depend on the specific system and input.

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CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The sensitivity of discrete systems has been investigated and discrete sensitivity equations derived for parameter and sampling interval variations. The effects of parameter variations were studied by two methods. The first approach used a perturbation matrix to change the system parameters. In the perturbation matrix approach, a discrete equation was derived for the exact change in the state vector due to parameter variations. Also, an equation was derived that ignored second and higher ordered effects. The only difference in the two equations was a single term. Both equations are quite simple and are well suited for computation in sequence with the system equations.

There are several restrictions on the perturbation matrix method. It is not applicable for nonlinear systems. Also, only perturbations in the system matrix, A, are included, and no provisions have been made to vary elements of the distribution matrix, B. This restriction could be removed with additional work. Another limitation,

which is less restrictive, is the inability to generalize the equation for the exact change. In Chapter 3, the equation for first-order effects was generalized by writing it in terms of the ratio of the change in the state vector to the perturbation. This cannot be done for the exact variation. However, this is not a serious limitation, since the first-order equation was found to be accurate enough for most parameter perturbation studies.

Parameter variations in discrete systems were also studied with the classical (sensitivity vector) approach. This method, based on partial derivatives, is more general than the perturbation matrix method since it can be extended to nonlinear systems. A discrete sensitivity equation was derived for the first-order effects on the state variable due to variations in the system parameters. The sensitivity equation is a linear, time-varying difference equation and is easily solved on a digital computer. The perturbation matrix method and the sensitivity vector approach gave identical results for first-order effects. This is significant since different methods were used to derive the equations.

The sensitivity of the state vector to variations in sampling interval was investigated on a global and local basis. For global sensitivity, all sampling intervals are equal and are considered to have the same variations. Since it gives the effect of a change in T on the state vector, global sensitivity can be used to extrapolate

performance about a given sampling rate. It could also be used to optimize the system sampling rate. It should be noted that the accuracy of the extrapolation deteriorates when the product of the change in sampling interval and the number of samples is greater than a sampling interval.

The local sensitivity function gives the effect on the state vector of a change in a sampling interval. Both local and global sensitivity functions can be calculated from linear, time-varying difference equations. For a linear system, the effects of global changes in sampling intervals can be determined from the sum of the local effects. Also, local sensitivity can be used to extrapolate performance about a single sampling interval.

Local sampling interval sensitivity was used to implement variable increment optimal sampling. A performance index was incorporated into sampling interval calculations by means of a straight-line approximation for the response over a single interval. Several problems were encountered with the calculation procedure. In order to effectively alter the sampling rate, the state of the system must be sensitive to changes in sampling interval, and some provision must be made to initiate sampling immediately after sudden changes in the input. Also, a predictor-corrector scheme for sensitivity was found desirable. Particular attention should be given to the limits on sampling interval size since the system

will become unstable if the sampling interval is not limited.

Recommendations

In the course of this research project, a number of problems were encountered that were not essential parts of the research goals and time limitations prevented subsequent study. Hopefully, the solution of these problems might improve the techniques that have been developed. Recommendations for future study will be listed as follows:

- Investigate analytically the stability of the sampling interval calculation loop.
- Include more parameters in the perturbation matrix approach.
- Investigate the use of variable increment sampling in mathematical modeling of systems of higher dimensionality and complexity than those considered here.
- Incorporate variable increment sampling into an on-line optimal-adaptive control scheme.

BIOGRAPHICAL SKETCH

Archie Wayne Bennett was born May 5, 1937, in Rocky Mount, Virginia. In June, 1955, he was graduated from Franklin County High School. He entered Virginia Polytechnic Institute on the co-operative engineering program and was awarded the degree of Bachelor of Science in Electrical Engineering (with honors) in June, 1960.

He went to work for the Industry Control Department of the General Electric Company in June, 1960, and also enrolled in the graduate school of Virginia Polytechnic Institute. In September, 1961, he became an Instructor of Electrical Engineering at Virginia Polytechnic Institute. In June, 1963, he received the degree of Master of Science in Electrical Engineering. He took a leave of absence in September, 1964, and until the present time has pursued a program leading to the degree of Doctor of Philosophy at the University of Florida.

Archie Wayne Bennett is married to the former Shirley Ann Turner and is the father of one child. He is a registered Professional Engineer and a member of the Institute of Electrical and Electronics Engineers, Eta Kappa Nu, Tau Beta Pi, and Phi Kappa Phi.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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